NORMAL FAMILIES AND CONVERGENCE OF CONFORMAL MAPPINGS

By

Miss Kamonrat Kamjornkittikoon

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics

Department of Mathematics

Graduate School, Silpakorn University

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วงศ์ปรกติและการลู่เข้าของการส่งคงแบบ

โดย

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตร์มหาบัณฑิต

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The Graduate School, Silpakorn University, has approved and accredited the Thesis title of "Normal families and convergence of conformal mappings" submitted by Miss Kamonrat Kamjornkittikoon as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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KAMONRAT KAMJORNKITTIKOON : NORMAL FAMILIES AND CONVERGENCE OF CONFORMAL MAPPINGS. THESIS ADVISORS : MALINEE CHAIYA,Ph.D., AND PROF.RAIMO NÄKKI,Ph.D. 42 pp.

One of the most important results in Complex Analysis is Montel's theorem which states that a family $\mathscr{F}$ of analytic functions $f : D \to \mathbb{R}^2$, where D is a domain in $\mathbb{R}^2$, is a normal family if and only if $\mathscr{F}$ is locally bounded.

In this thesis we study Montel's Theorem in the context of conformal and quasiconformal mappings. For instance, we show that a family $\mathscr{F}$ of conformal mappings is normal if each element f of $\mathscr{F}$ omits two values a_f and b_f whose distance is uniformly bounded from below by a positive number. Convergence of such mappings will also be investigated. Finally, extensions to quasiconformal mappings will be indicated.

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คำสำคัญ : การส่งคงแบบ/การส่งกึ่งคงแบบ/วงศ์ปรกติ

กมลรัตน์ กำจรกิตติคุณ : วงศ์ปรกติและการลู่เข้าของการส่งคงแบบ. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : อ.ดร.มาลินี ชัยยะ และ Professor Dr.RAIMO NÄKKI. 42 หน้า.

ทฤษฎีบทของมอนเทลเป็นหนึ่งในทฤษฎีบทที่สำคัญมากในวิชาการวิเคราะห์เชิงซ้อนซึ่งกล่าวว่า ถ้า $\mathscr{F}$ เป็นวงศ์ของฟังก์ชันวิเคราะห์ $f : D \to \mathbb{R}^2$ เมื่อ D เป็นโดเมนใน $\mathbb{R}^2$ จะได้ว่า $\mathscr{F}$ เป็นวงศ์ปรกติ ก็ต่อเมื่อ $\mathscr{F}$ มีขอบเขตเฉพาะที่

ในวิทยานิพนธ์นี้ เราได้ศึกษาทฤษฎีบทของมอนเทลในบริบทของการส่งคงแบบและการส่งกึ่งคงแบบ ตัวอย่างเช่นเราแสดงว่าวงศ์ของการส่งคงแบบเป็นวงศ์ปรกติ ถ้าแต่ละสมาชิก f ใน $\mathscr{F}$ ไม่ส่งค่าไปยัง a_f และ b_f ซึ่งระยะทางระหว่าง a_f และ b_f มีขอบเขตล่างที่เป็นบวกแบบเอกรูป อีกทั้งศึกษาการลู่เข้าของแต่ละการส่ง นอกจากนี้เราได้ขยายแนวทางการศึกษาไปยังการส่งกึ่งคงแบบอีกด้วย

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Contents

Chapter 1

Introduction

Montel's theorem on normal families is one of the most important and useful results in the classical Complex Analysis. Normality of the family of mappings is a sort of compactness property. To wit, a family of mappings is a normal family if every sequence in the family has a pointwise convergent subsequence, the convergence being uniform on all compact subsets.

The celebrated theorem of Montel usually addresses analytic (or meromorphic) functions defined in a domain (open connected set) D in the complex plane. The theorem asserts that a family $\mathscr{F}$ of analytic functions $f : D \to \mathbb{R}^2$ is a normal family if and only if $\mathscr{F}$ is locally bounded. That is, $\mathscr{F}$ is a normal family precisely when, for every compact set $K \subset D$ there is a constant M such that $|f(z)| \leq M$ whenever $z \in K$ and $f \in \mathscr{F}$.

In this thesis we will establish conformal versions of Montel's theorem. In particular we will demonstrate that a family $\mathscr{F}$ of conformal mappings $f : D \to \overline{\mathbb{R}^2}$ is a normal family if each $f \in \mathscr{F}$ omits 2 values a_f and b_f with chordal distance $q(a_f, b_f) \geq r$, where $r > 0$ is independent of f.

We also investigate convergence of a sequence of conformal mappings. As our main result we will prove that if (f_n) is a sequence of conformal mappings of a domain D into $\overline{\mathbb{R}^2}$ which converges pointwise in D to a mapping f, then there are 3 possibilities for f:

(1) f is a constant mapping.

(2) f assumes exactly 2 values, one of which at exactly one point.

(3) f is a conformal mapping.

Extensions of the aforementioned results into the realm of quasiconformal mappings will be indicated in the final section of this thesis.

Chapter 2

Modulus of a Path Family

In this chapter we study the basic properties related to the modulus of a path family and present examples. Also, we introduce the concept of a ring and derive modulus estimates in the spherical metric. The modulus of a path family is our main tool in the investigation of conformal and quasiconformal mappings.

2.1 The modulus

A **path** γ is a continuous mapping of an interval in $\mathbb{R}$ into $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$. A path is **closed** if the interval is closed, and **open** if the interval is open. The **locus** $|\gamma|$ of a path γ is the range of γ. A **subpath** of a path γ is a restriction of γ to a subinterval.

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed path and let $a = t_0 < t_1 < \ldots < t_k = b$ be a subdivision of $[a,b]$. The supremum of the sums

$$\sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|$$

over all subdivisions is called the **length** of γ and denoted by $\ell(\gamma)$. Thus $0 \leq \ell(\gamma) \leq \infty$ and $\ell(\gamma) = 0$ if and only if γ is constant.

A path γ is **rectifiable** if $\ell(\gamma) \neq \infty$, otherwise **non-rectifiable**. Also a path in $\overline{\mathbb{R}^2}$ such that $\infty \in \gamma([a,b])$ is non-rectifiable, except for the constant path $\gamma(t) \equiv \infty$, for which we define $\ell(\gamma) = 0$. A path γ is **locally rectifiable** if every closed subpath is rectifiable. Let γ be a rectifiable path. The function $s : [a,b] \to \overline{\mathbb{R}^2}$ defined by $s(t) = \ell(\gamma|[a,t])$ is called the **length-function**.

A path $\gamma : [a,b] \to \mathbb{R}^2$ is obtained from a path $\beta : [c,d] \to \mathbb{R}^2$ by an **increasing change of parameter** if there exists an increasing continuous mapping h of $[a,b]$ onto $[c,d]$ such that $\gamma = \beta \circ h$.

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a rectifiable path. The path $\gamma^0 : [0,\ell(\gamma)] \to \mathbb{R}^2$ is called the **normal representation** of γ if it has the following properties:

1. γ is obtained from γ^0 by an increasing change of parameter.
2. $\ell(\gamma^0|[0,t]) = t$ for all $0 \le t \le \ell(\gamma)$.

Let A be a Borel set (see Appendix) in $\mathbb{R}^2$ and let $\rho : A \to [0,\infty]$ be a Borel function (see Appendix). For each rectifiable closed path $\gamma : [a,b] \to A$, we define the **line integral** of ρ over γ as follows:

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} \rho\left(\gamma^0(t)\right) dt.$$

We also use the notation

$$\int_\gamma \rho ds = \int_\gamma \rho(z)|dz|.$$

Definition 2.1.1. *Let Γ be a family of paths in $\overline{\mathbb{R}^2}$. Consider all Borel functions $\rho : \mathbb{R}^2 \to [0,\infty]$ such that*

$$\int_\gamma \rho ds \ge 1$$

for every locally rectifiable path $\gamma \in \Gamma$. Denote this collection by $\mathscr{F}(\Gamma)$ and call such $\rho \in \mathscr{F}(\Gamma)$ ***admissible*** *for Γ. For each $p \ge 1$ we set*

$$M_p(\Gamma) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int_{\mathbb{R}^2} \rho^p dm,$$

where m is the Lebesgue measure in $\mathbb{R}^2$. If $\mathscr{F}(\Gamma) = \phi$, we define $M_p(\Gamma) = \infty$. Clearly $0 \le M_p(\Gamma) \le \infty$.

The number $M_p(\Gamma)$ is called the ***p-modulus*** *of Γ. The most important case in the plane $\overline{\mathbb{R}^2}$ is the case $p = 2$. We shall denote $M_2(\Gamma)$ by $M(\Gamma)$ and call it the* ***modulus*** *of Γ.*

Theorem 2.1.2. *M_p is an outer measure in the collection of all path families:*

(1) $M_p(\phi) = 0$,

(2) $\Gamma_1 \subset \Gamma_2$ *implies* $M_p(\Gamma_1) \le M_p(\Gamma_2)$,

(3) $M_p\left(\bigcup_{k=1}^{\infty} \Gamma_k\right) \le \sum_{k=1}^{\infty} M_p(\Gamma_k)$.

Proof. (1) The function $\rho : \mathbb{R}^2 \to [0,\infty]$, defined by $\rho(z) \equiv 0$ for all $z \in \mathbb{R}^2$, is admissible for the path family ϕ. Hence

$$M_p(\phi) \le \int_{\mathbb{R}^2} 0^p dm = 0,$$

and so $M_p(\phi) = 0$.

(2) Every $\rho \in \mathscr{F}(\Gamma_2)$ is also in $\mathscr{F}(\Gamma_1)$. Hence

$$M_p(\Gamma_1) \leq \int_{\mathbb{R}^2} \rho^p dm.$$

Thus

$$M_p(\Gamma_1) \leq \inf_{\rho \in \mathscr{F}(\Gamma_2)} \int_{\mathbb{R}^2} \rho^p dm = M_p(\Gamma_2).$$

(3) We may assume that each $M_p(\Gamma_k) < \infty$. Fix $\epsilon > 0$. For each k, we choose $\rho_k \in \mathscr{F}(\Gamma_k)$ so that

$$\int_{\mathbb{R}^2} \rho_k^p dm < M_p(\Gamma_k) + \frac{\epsilon}{2^k}.$$

The function ρ, defined by $\rho(z) = \left(\sum \rho_k^p(z)\right)^{1/p}$ for all $z \in \mathbb{R}^2$, is admissible for $\bigcup_{k=1}^{\infty} \Gamma_k$, because if $\gamma \in \bigcup_{k=1}^{\infty} \Gamma_k$, then $\gamma \in \Gamma_k$ for some k and

$$\int_\gamma \rho |dz| \geq \int_\gamma \rho_k |dz| \geq 1,$$

since $\rho \geq \rho_k$ for each k. Consequently,

$$\begin{aligned} M_p\left(\bigcup_{k=1}^{\infty} \Gamma_k\right) &\leq \int_{\mathbb{R}^2} \rho^p dm = \int_{\mathbb{R}^2} \left(\sum_{k=1}^{\infty} \rho_k^p\right) dm \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} \rho_k^p dm \leq \sum_{k=1}^{\infty} \left(M_p(\Gamma_k) + \frac{\epsilon}{2^k}\right) \\ &= \sum_{k=1}^{\infty} M_p(\Gamma_k) + \epsilon. \end{aligned}$$

Hence

$$M_p\left(\bigcup_{k=1}^{\infty} \Gamma_k\right) \leq \sum_{k=1}^{\infty} M_p(\Gamma_k).$$

□

Definition 2.1.3. *Let Γ_1 and Γ_2 be two families of paths. We say that Γ_1* ***minorizes*** *Γ_2 and denote $\Gamma_1 < \Gamma_2$ if every path $\gamma \in \Gamma_2$ has a subpath in Γ_1.*

Theorem 2.1.4. *If $\Gamma_1 < \Gamma_2$, then $M_p(\Gamma_1) \geq M_p(\Gamma_2)$.*

Proof. Suppose that $\Gamma_1 < \Gamma_2$. We claim that $\mathscr{F}(\Gamma_1) \subset \mathscr{F}(\Gamma_2)$. Let $\rho \in \mathscr{F}(\Gamma_1)$. We want to show that $\rho \in \mathscr{F}(\Gamma_2)$. Take an arbitrary locally rectifiable path $\gamma \in \Gamma_2$. Then γ has a subpath $\tilde{\gamma} \in \Gamma_1$. Thus

$$\int_{\tilde{\gamma}} \rho |dz| \geq 1.$$

Hence

$$\int_{\gamma} \rho |dz| \geq \int_{\tilde{\gamma}} \rho |dz| \geq 1$$

and so $\rho \in \mathscr{F}(\Gamma_2)$. Now since

$$\int_{\mathbb{R}^2} \rho^p dm \geq M_p(\Gamma_2),$$

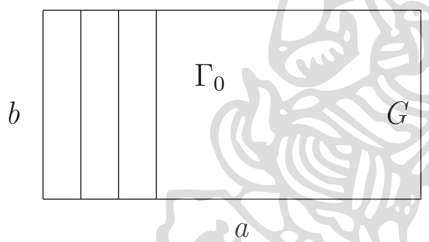
we obtain, taking the infimum over all such $\rho \in \mathscr{F}(\Gamma_1)$, that

$$M_p(\Gamma_1) = \inf_{\rho \in \mathscr{F}(\Gamma_1)} \int_{\mathbb{R}^2} \rho^p dm \geq M_p(\Gamma_2).$$

□

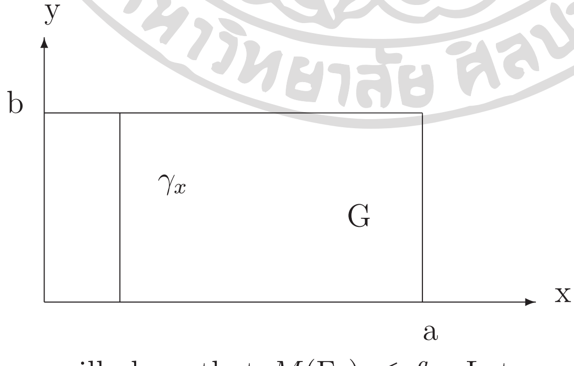
For a given path family Γ, it is very difficult (in general impossible) to compute $M(\Gamma)$. The modulus of Γ is known only for very few families Γ. We next give some examples:

Example 1. *Let G be a bounded rectangular domain in $\mathbb{R}^2$ whose width and height are a and b respectively. Let Γ_0 be the collection of all vertical line segments in G. Then*



$$M(\Gamma_0) = \frac{a}{b}.$$

Proof.



First, we will show that $M(\Gamma_0) \leq \frac{a}{b}$. Let us construct $\rho \in \mathscr{F}(\Gamma_0)$ in such a way that

$$\int_{\mathbb{R}^2} \rho^2 dm = \frac{a}{b}.$$

Set

$$\rho_0(z) = \begin{cases} \frac{1}{b} & \text{if } z \in G, \\ 0 & \text{if } z \notin G. \end{cases}$$

Clearly,

$$\int_\gamma \rho_0 |dz| = \int_\gamma \frac{1}{b}|dz| = \frac{1}{b}\int_\gamma |dz| \geq 1$$

for all $\gamma \in \Gamma_0$. Hence $\rho_0 \in \mathscr{F}(\Gamma_0)$ and therefore,

$$\begin{aligned} M(\Gamma_0) &\leq \int_{\mathbb{R}^2} \rho_0^2 dm = \int_G \rho_0^2 dm = \frac{1}{b^2}\int_G dm \\ &= \frac{1}{b^2}\text{area}(G) = \frac{ab}{b^2} = \frac{a}{b}. \end{aligned}$$

Next, we want to show that $M(\Gamma_0) \geq \frac{a}{b}$. Let $\rho \in \mathscr{F}(\Gamma_0)$. Then

$$\int_{\gamma_x} \rho |dz| = \int_0^b \rho(x+iy)dy \geq 1,$$

where γ_x is the vertical line segment in Γ_0 starting at $x \in [0,a]$. We need the following so-called Cauchy-Schwarz inequality for measurable functions f and g,

$$\left(\int_c^d f(t)g(t)dt\right)^2 \leq \int_c^d f(t)^2 dt \int_c^d g(t)^2 dt.$$

This is a special case of Hölder's inequality: if f and g are non-negative measurable functions with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_c^d f(t)g(t)dt \leq \left(\int_c^d f(t)^p dt\right)^{1/p}\left(\int_c^d g(t)^q dt\right)^{1/q}.$$

The Cauchy-Schwarz inequality is obtained when $p=q=2$. Now,

$$\begin{aligned} 1 \leq \left(\int_0^b \rho dy\right)^2 &= \left(\int_0^b \rho\cdot 1 dy\right)^2 \\ &\leq \int_0^b \rho^2 dy \int_0^b 1^2 dy \\ &= b\int_0^b \rho^2 dy. \end{aligned}$$

Hence

$$\frac{1}{b} \leq \int_0^b \rho^2(x+iy)dy.$$

Integrating over all $x \in [0,a]$, we obtain

$$\begin{aligned} \frac{a}{b} = \int_0^a \frac{1}{b}dx &\leq \int_0^a \left(\int_0^b \rho^2 dy\right)dx \\ &= \int_G \rho^2 dm \\ &\leq \int_{\mathbb{R}^2} \rho^2 dm. \end{aligned}$$

This is true for all $\rho \in \mathscr{F}(\Gamma_0)$. Hence it is true for the infimum to yield

$$\frac{a}{b} \leq \inf_{\rho \in \mathscr{F}(\Gamma_0)} \int \rho^2 dm = M(\Gamma_0).$$

Consequently,

$$M(\Gamma_0) = \frac{a}{b}.$$

□

Example 2. *Let A be a spherical annulus $B(0,b) \setminus \overline{B(0,a)}$ where $0 < a < b$. Let Γ_0 be the collection of all radial segments in A. Then*

$$M(\Gamma_0) = \frac{2\pi}{\log \frac{b}{a}}.$$

Proof. We will first show that $M(\Gamma_0) \leq \frac{2\pi}{\log \frac{b}{a}}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{|z| \log \frac{b}{a}} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

Given $\gamma_0 \in \Gamma_0$, we get

$$\int_{\gamma_0} \rho_0 |dz| = \int_{\gamma_0} \frac{1}{|z| \log \frac{b}{a}} |dz| = \int_a^b \frac{1}{r \log \frac{b}{a}} dr = 1.$$

Hence

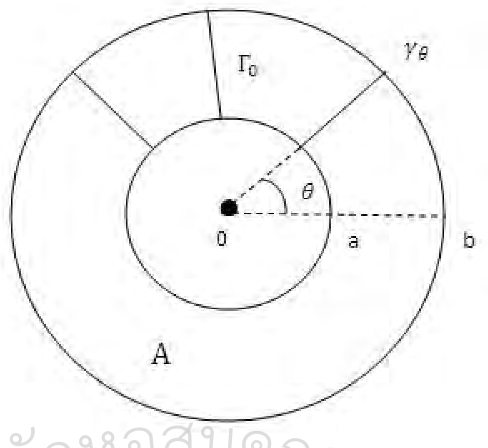
$$\begin{aligned} M(\Gamma_0) &\leq \int_{\mathbb{R}^2} \rho_0^2 dm = \int_A \rho_0^2 dm \\ &= \int_A \frac{1}{|z|^2 (\log \frac{b}{a})^2} dm \\ &= \int_0^{2\pi} \int_a^b \frac{1}{r^2 (\log \frac{b}{a})^2} r dr d\theta \\ &= \frac{2\pi}{\log \frac{b}{a}}. \end{aligned}$$

Next, we will prove that $M(\Gamma_0) \geq \frac{2\pi}{\log \frac{b}{a}}$. Let $\rho \in \mathscr{F}(\Gamma_0)$. For each $\theta \in [0, 2\pi]$, let $\gamma_\theta : [a,b] \to A$ be the line segment in Γ_0 defined by

$$\gamma_\theta(r) = re^{i\theta}.$$

Then

$$\int_{\gamma_\theta} \rho |dz| = \int_a^b \rho(re^{i\theta}) dr \geq 1.$$



Hence, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}1 &\leq \left(\int_a^b \rho dr\right)^2 = \left(\int_a^b \rho r^{1/2} \cdot r^{-1/2} dr\right)^2 \\ &\leq \int_a^b (\rho r^{1/2})^2 dr \int_a^b (r^{-1/2})^2 dr \\ &= \int_a^b \rho^2 r dr \int_a^b \frac{1}{r} dr \\ &= \log\frac{b}{a}\int_a^b \rho^2 r dr.\end{aligned}$$

Therefore

$$\frac{1}{\log\frac{b}{a}} \leq \int_a^b \rho^2 r dr.$$

Integrating over all $\theta \in [0, 2\pi]$, we get

$$\int_0^{2\pi} \frac{1}{\log\frac{b}{a}} d\theta \leq \int_0^{2\pi} \left(\int_a^b \rho^2 r dr\right) d\theta = \int_A \rho^2 dm \leq \int_{\mathbb{R}^2} \rho^2 dm.$$

That is,

$$\frac{2\pi}{\log\frac{b}{a}} \leq \int_{\mathbb{R}^2} \rho^2 dm$$

for all $\rho \in \mathscr{F}(\Gamma_0)$. Hence

$$\frac{2\pi}{\log\frac{b}{a}} \leq \inf_{\rho\in\mathscr{F}(\Gamma_0)} \int_{\mathbb{R}^2} \rho^2 dm = M(\Gamma_0).$$

Consequently,

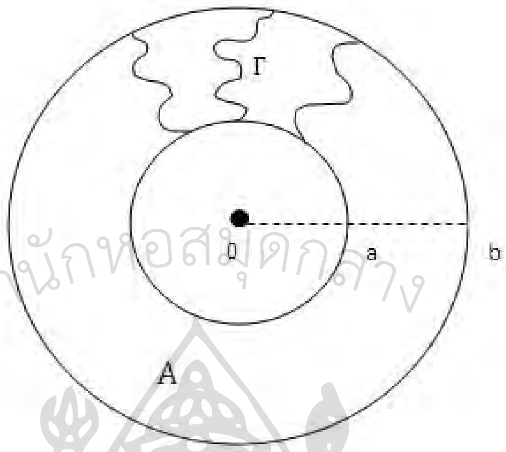
$$M(\Gamma_0) = \frac{2\pi}{\log\frac{b}{a}},$$

as desired.

$\square$

Example 3. *Let A be a spherical annulus $B(0,b) \setminus \overline{B(0,a)}$ where $0 < a < b$, and let Γ be the collection of all paths in A joining the boundary components of A. Then*

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$



Proof. We will show that $M(\Gamma) \leq \frac{2\pi}{\log \frac{b}{a}}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{|z| \log \frac{b}{a}} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

The function ρ_0 is admissible for Γ, because if $\gamma \in \Gamma$ is locally rectifiable, then

$$\int_\gamma \rho_0 |dz| \geq \int_a^b \rho_0 dr = 1.$$

Thus

$$M(\Gamma) \;\; \leq \;\; \int_A \rho_0^2 dm = \frac{2\pi}{\log \frac{b}{a}}.$$

Next, we will prove that $M(\Gamma) \geq \frac{2\pi}{\log \frac{b}{a}}$. Let Γ_0 be the collection of all radial segments in A. Then $\Gamma_0 \subset \Gamma$. By Theorem 2.1.2, $M(\Gamma_0) \leq M(\Gamma)$. Therefore

$$\frac{2\pi}{\log \frac{b}{a}} \leq M(\Gamma).$$

Consequently,

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$

□

Example 4. *Let A be the spherical annulus $B(0,b) \setminus \{0\}$ where $b > 0$, and let Γ be the collection of all paths in A joining the boundary components of A. Then*

$$M(\Gamma) = 0.$$

Proof. Let B be a spherical annulus $B(0,b) \setminus \overline{B(0,a)}$ where $0 < a < b$, and let Γ_B be the collection of all paths in B joining the boundary components of B. Then $\Gamma_B < \Gamma$. By Theorem 2.1.4, we have

$$M(\Gamma_B) \geq M(\Gamma).$$

Note that $M(\Gamma_B) = \frac{2\pi}{\log \frac{b}{a}}$ by Example 3. Thus

$$\frac{2\pi}{\log \frac{b}{a}} \geq M(\Gamma).$$

Since $\frac{2\pi}{\log \frac{b}{a}} \to 0$ as $a \to 0$, we get $M(\Gamma) = 0$ as desired.

□

2.2 Conformal Mappings

Let D and D' be domains in $\mathbb{R}^2$. Recall that a mapping $f : D \to D'$ is a **homeomorphism** if f is bijective and if both f and f^{-1} are continuous.

Let C^1 be the set of all continuous mapppings in $\mathbb{R}^2$ that are differentiable at every point in the domain and their partial derivatives are continuous.

Next let $f : D \to \mathbb{R}^2$ be a C^1-mapping where $f(z) = (u(z), v(z))$. We write $f = u + iv$ and $z = x + iy$ where $x, y \in \mathbb{R}$. Define **the Jacobian of f at z** as

$$\begin{aligned} J(z,f) &= \begin{vmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{vmatrix} \\ &= u_x(z)v_y(z) - u_y(z)v_x(z). \end{aligned}$$

If $J(z,f) \neq 0$ for all z, then f is called a **diffeomorphism**. Since $J(z,f)$ is continuous and D is connected, we see that $J(z,f) > 0$ for all $z \in D$ or $J(z,f) < 0$ for all $z \in D$.

Let $z, w \in \mathbb{R}^2$ be non-zero. We call

$$\mathrm{Arg}\left(\frac{z}{w}\right)$$

the **oriented angle** from z to w and denote it by $\theta(z,w)$. Then $-\pi < \theta(z,w) \leq \pi$.

Definition 2.2.1. *Let D be a domain in $\mathbb{R}^2$. A diffeomorphism $f : D \to \mathbb{R}^2$ is said to be **conformal** at a point z_0 if*

$$\theta(\alpha, \beta) = \theta\left(f(\alpha), f(\beta)\right)$$

*for all smooth paths α, β emanating at z_0. If this is true at every point of D, the mapping f is said to be **conformal** in D. If D is a domain in $\overline{\mathbb{R}^2}$ and if $f : D \to \overline{\mathbb{R}^2}$ is a homeomorphism, then it is said to be conformal in D provided $f|D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.*

Conformality can be characterized in terms of complex derivative:

Theorem 2.2.2. *Let D be a domain in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}^2$ be a diffeomorphism. The following conditions are equivalent:*

(1) *f is conformal at z_0.*

(2) *f has a complex derivative $f^{'}(z_0)$.*

Proof. See [4, Theorem 1.1, page 380]. □

Theorem 2.2.3. *Let D be a domain in $\mathbb{R}^2$. A mapping $f : D \to \mathbb{R}^2$ is a conformal mapping of D if and only if f is injective and analytic.*

Proof. Suppose that f is a conformal mapping of D. By definition, f is a diffeomorphism. That is, f is injective. By Theorem 2.2.2, f is differentiable at each point of D. Thus f is analytic in D.

Conversely, assume that f is injective and analytic. Then f is an open bijective mapping onto $f(D)$. By a result in Topology, the inverse map f^{-1} is also continuous, therefore f is a homeomorphism. Since an analytic map is a C^∞-function, f is a diffeomorphism. Therefore, by Theorem 2.2.2, f is a conformal mapping. □

Lemma 2.2.4. *Let U be an open set in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}^2$ be continuous. Suppose that $\gamma : [a, b] \to U$ is a locally rectifiable path such that f is absolutely continuous on every closed subpath of γ. Then $f \circ \gamma$ is locally rectifiable. If $\rho : |f \circ \gamma| \to [0, \infty]$ is a Borel function, then*

$$\int_{f\circ\gamma} \rho ds \le \int_{\gamma} \rho(f(z)) L(z, f) |dz|,$$

where $L(x, f) = \limsup_{|h|\to 0} \frac{|f(x+h)-f(x)|}{|h|}$. Furthermore the equality holds if $\lim_{|h|\to 0} \frac{|f(x+h)-f(x)|}{|h|}$ exists.

Proof. See [5, Theorem 5.3, page 12]. □

Theorem 2.2.5. *Let D and D' be domains in $\mathbb{R}^2$, let $\rho : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be a non-negative Borel function, let $f : D \to D'$ be a conformal mapping, and let γ be a locally rectifiable path in D. Then*

$$\int_{f\circ\gamma} \rho ds = \int_{\gamma} \rho(f(z))|f'(z)||dz|.$$

Proof. Since f is conformal, f is differentiable in D. Thus $L(x, f) = |f'(x)|$ for every $x \in D$. Since $\lim_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}$ exists for every $x \in |\gamma|$, there is, in fact, equality in Lemma 2.2.4. Thus

$$\int_{f\circ\gamma} \rho ds = \int_{\gamma} \rho(f(x))L(x, f)|dx| = \int_{\gamma} \rho(f(x))|f'(x)||dx|.$$

□

Suppose that A is a subset of $\overline{\mathbb{R}^2}$ and that $f : A \to \overline{\mathbb{R}^2}$ is continuous. If Γ is a family of paths in A, then the family $\Gamma' = \{f \circ \gamma | \gamma \in \Gamma\}$ is called the **image** of Γ under f.

Theorem 2.2.6. *If $f : D \to D'$ is a conformal mapping, then $M(\Gamma') = M(\Gamma)$ for every path family Γ in D.*

Proof. First, we will show that $M(\Gamma) \le M(\Gamma')$. Let $\hat{\rho} \in \mathscr{F}(\Gamma')$. Set

$$\rho(z) = \begin{cases} \hat{\rho}(f(z))|f'(z)| & \text{if } z \in D, \\ 0 & \text{if } z \notin D. \end{cases}$$

Since, by Theorem 2.2.5,

$$\begin{aligned} \int_{\gamma} \rho|dz| &= \int_{\gamma} \hat{\rho}(f(z))|f'(z)||dz| \\ &= \int_{f\circ\gamma} \hat{\rho}(z)|dz| \ge 1, \end{aligned}$$

for every locally rectifiable path γ in Γ, we have $\rho \in \mathscr{F}(\Gamma)$. Thus

$$\begin{aligned} M(\Gamma) &\le \int_{\mathbb{R}^2} \rho^2 dm = \int_{D} \hat{\rho}(f(z))^2 |f'(z)|^2 dm \\ &= \int_{f(D)} \hat{\rho}^2 dm \le \int_{\mathbb{R}^2} \hat{\rho}^2 dm. \end{aligned}$$

Because $\hat{\rho} \in \mathscr{F}(\Gamma')$ is arbitrary,

$$M(\Gamma) \le \inf_{\hat{\rho}\in\mathscr{F}(\Gamma')} \int_{\mathbb{R}^2} \hat{\rho}^2 dm = M(\Gamma').$$

Next, since f is conformal, f^{-1} is also conformal. Then the reverse inequality can be proved by applying a similar argument with f^{-1} in place of f.

□

2.3 Rings

A domain $A \subset \overline{\mathbb{R}^2}$ is a **ring** if A^c, the complement of A, has exactly 2 components. If the components of A^c are C_0 and C_1, we denote $A = R(C_0, C_1)$.

By Topology, ∂A, the boundary of A, has also two components, namely $B_0 = C_0 \cap \overline{A}$ and $B_1 = C_1 \cap \overline{A}$, where $\overline{A}$ is the closure of A. For each ring $A = R(C_0, C_1)$, let Γ_A be the family of all closed paths that join B_0 and B_1 in A.

Theorem 2.3.1. *If $A = R(C_0, C_1)$ and $A' = R(C_0', C_1')$ are rings such that $C_i \subset C_i'$, then $M(\Gamma_A) \le M(\Gamma_{A'})$.*

Proof. Let Γ_A^2 be the family of all closed paths which join C_0 and C_1 in $\overline{\mathbb{R}^2}$ and $\Gamma_{A'}^2$ be the family of all closed paths which join C_0' and C_1' in $\overline{\mathbb{R}^2}$. Because $\Gamma_A \subset \Gamma_A^2 > \Gamma_A$, we have $M(\Gamma_A) = M(\Gamma_A^2)$. Since $\Gamma_{A'}^2 < \Gamma_A^2$, we obtain

$$M(\Gamma_A) = M(\Gamma_A^2) \le M(\Gamma_{A'}^2) = M(\Gamma_{A'}).$$

□

Theorem 2.3.2. *If A is a ring, then $M(\Gamma_A)$ is finite.*

Proof. Let $A = R(C_0, C_1)$ be a ring. We may assume that C_0 is bounded. Let $h = \inf\{|z - w| : z \in C_0, w \in C_1\}$ and let $d(A, z) = \inf\{|z - w| : w \in A\}$. Then $h > 0$. Set $E = \{z \in \mathbb{R}^2 : d(C_0, z) \le h\}$. Define $\rho : \mathbb{R}^2 \to \mathbb{R}$ by

$$\rho(z) = \begin{cases} \frac{1}{h} & \text{if } z \in E, \\ 0 & \text{if } z \notin E. \end{cases}$$

Let $\gamma \in \Gamma_A$. Then

$$\int_\gamma \rho|dz| = \int_{\gamma \cap E} \frac{1}{h}|dz| = \frac{1}{h}\int_{\gamma \cap E} |dz| \ge 1.$$

Thus $\rho \in \mathscr{F}(\Gamma_A)$. Hence

$$M(\Gamma_A) \le \int \rho^2 dm = \frac{m(E)}{h^2} < \infty.$$

□

Definition 2.3.3. *Given $r > 0$, let $\Phi(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^2}$ with the following properties:* (1) *C_0 contains the origin and a point a such that $|a| = 1$,* (2) *C_1 contains ∞ and a point b such that $|b| = r$. We denote*

$$\xi(r) = \inf_{A \in \Phi(r)} M(\Gamma_A).$$

Remark 2.3.4. *By Theorem 2.3.2, $\xi(r)$ is a non-negative finite number.*

Theorem 2.3.5. *The function $\xi : (0, \infty) \to \mathbb{R}$ has the following properties:*

(1) *ξ is decreasing.*

(2) *$lim_{r\to\infty}\xi(r) = 0$.*

(3) *$lim_{r\to 0}\xi(r) = \infty$.*

(4) *$\xi(r) > 0$ for every $r > 0$.*

Proof. See [5, Theorem 11.7, page 34]. □

Theorem 2.3.6. *Suppose that $A = R(C_0, C_1)$ is a ring and that $a, b \in C_0$ and $c, \infty \in C_1$. Then*

$$M(\Gamma_A) \geq \xi\left(\frac{|c-a|}{|b-a|}\right).$$

Proof. Let f be the function defined by

$$f(z) = \frac{z-a}{b-a}$$

for all $z \in \overline{\mathbb{R}^2}$. Then f is conformal and

$$f(a) = 0, f(b) = 1, f(c) = \frac{c-a}{b-a} \quad \text{and} \quad f(\infty) = \infty.$$

Moreover, the ring $f(A) = R\left(C_0', C_1'\right)$ has the following properties:
(1) C_0' contains the origin and a point $f(b)$ such that $|f(b)| = 1$,
(2) C_1' contains ∞ and a point $f(c)$ such that $|f(c)| = \dfrac{|c-a|}{|b-a|}$.
By Theorem 2.2.6 and the definition of ξ, we get

$$M(\Gamma_A) = M(\Gamma_{f(A)}) \geq \xi\left(\frac{|c-a|}{|b-a|}\right).$$

□

Theorem 2.3.7. *Suppose that $A = R(C_0, C_1)$ is a ring. Then $M(\Gamma_A) = 0$ if and only if C_0 or C_1 consists of a single point.*

Proof. Without loss of generality, we may assume that C_0 consists of a single point, namely a. Let $\epsilon = q(C_0, C_1)$ and $R = B(a, \frac{\epsilon}{2}) \setminus \{a\}$. Then R is a ring such that $\Gamma_R < \Gamma_A$. Thus $M(\Gamma_A) \leq M(\Gamma_R) = 0$ by Example 4. Therefore $M(\Gamma_A) = 0$. Conversely, suppose that $M(\Gamma_A) = 0$. We want to prove that C_0 or C_1 consists of a single point. Assume that neither C_0 nor C_1 consists of a single point. Then we can choose distinct points $a, b \in C_0$ and $c, d \in C_1$. By performing a Mobious transformation $f(z) = \frac{1}{z-d}$, we may assume that $d = \infty$. By Theorem 2.3.6 and Theorem 2.3.5(4), we get

$$M(\Gamma_A) \geq \xi\left(\frac{|c-a|}{|b-a|}\right) > 0$$

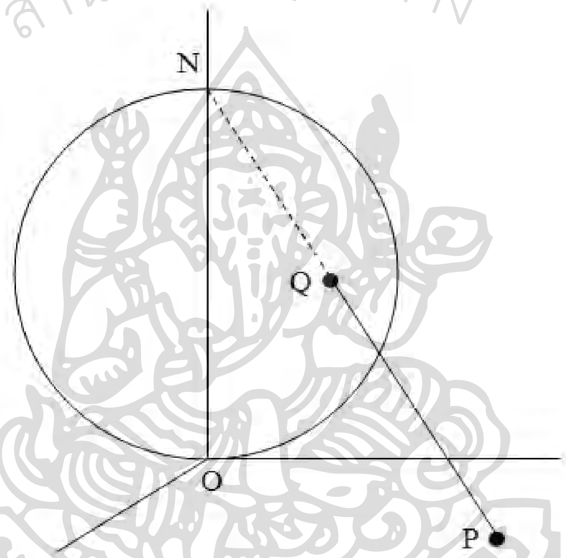
which is a contradiction. Hence the proof is complete. □

2.4 Spherical metric

We denote $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$, the one point compactification of $\mathbb{R}^2$. A line in $\overline{\mathbb{R}^2}$ is defined to be $L \cup \{\infty\}$, where L is a line in $\mathbb{R}^2$. The plane will be equipped with the topology induced by the chordal metric obtained from the standard stereographic projection. To define the chordal metric, let S be the sphere in $\mathbb{R}^3$ of radius $1/2$ centered at $(0,0,1/2)$ and let $N = (0,0,1)$ be the north pole on S. The **stereographic projection** of $\overline{\mathbb{R}^2}$ on S is the mapping f defined by, for $P \in \overline{\mathbb{R}^2}$,

$$f(P) = \begin{cases} N & \text{if } P = \infty, \\ Q & \text{if } P \neq \infty, \end{cases}$$

where $Q = S \cap \overline{NP}$ and $\overline{NP}$ is the line from N to P.



To define the chordal metric, let A be a point in $\mathbb{R}^2$ and let $f(A) = a$ where $f : \overline{\mathbb{R}^2} \to S$ is the stereographic projection.

The triangles NOA and NaO are similar right triangles. Thus

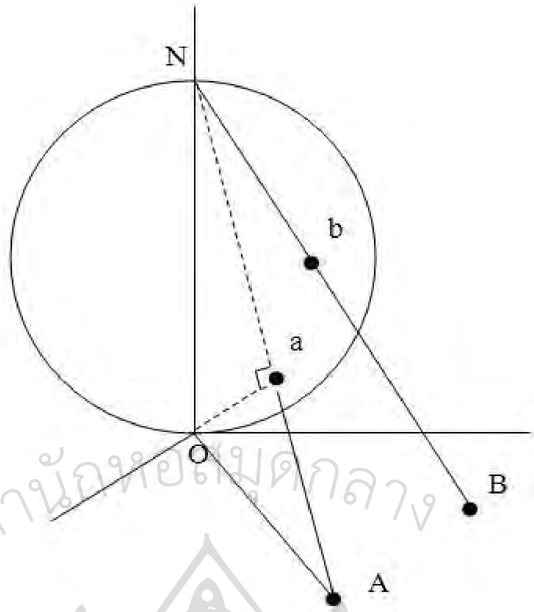
$$\frac{aN}{ON} = \frac{ON}{AN}.$$

That is,

$$|a - N| = \frac{1}{\sqrt{1 + |A|^2}}.$$

Let B be another point in $\mathbb{R}^2$ and let $f(B) = b$. Then

$$|b - N| = \frac{1}{\sqrt{1 + |B|^2}}.$$



Hence

$$\frac{aN}{Nb} = \frac{1/\sqrt{1+|A|^2}}{1/\sqrt{1+|B|^2}} = \frac{\sqrt{1+|B|^2}}{\sqrt{1+|A|^2}} = \frac{NB}{NA}.$$

Thus the triangles NAB and Nba are similar and we obtain

$$\begin{aligned} \frac{aN}{ab} &= \frac{NB}{AB} \\ \frac{1/\sqrt{1+|A|^2}}{|a-b|} &= \frac{1/\sqrt{1+|B|^2}}{|A-B|} \\ |a-b| &= \frac{|A-B|}{\sqrt{1+|A|^2}\sqrt{1+|B|^2}}. \end{aligned}$$

Now we define the **chordal metric** q in $\overline{\mathbb{R}^2}$ by

$$\begin{aligned} q(A,B) &= \frac{|A-B|}{\sqrt{1+|A|^2}\sqrt{1+|B|^2}}, \text{ if } A \neq \infty \neq B, \\ q(A,\infty) &= \frac{1}{\sqrt{1+|A|^2}}, \text{ if } A \neq \infty, \text{ and} \\ q(\infty,\infty) &= 0. \end{aligned}$$

Let $E \subset \overline{\mathbb{R}^2}$ be non-empty. We define the diameter of E by

$$q(E) = \sup\{q(a,b) | a, b \in E\}.$$

If E and F are non-empty subsets of $\overline{\mathbb{R}^2}$, then the distance from E to F, denoted by $q(E,F)$, is defined to be

$$q(E,F) = \inf\{q(a,b) | a \in E, b \in F\}.$$

2.5 Modulus estimates in the spherical metric

Definition 2.5.1. *Given* $0 < r \le 1$, *let* $\Psi(r)$ *be the set of all rings* $A = R(C_0, C_1)$ *in* $\overline{\mathbb{R}^2}$ *with the following properties:* (1) $q(C_0) \ge r$, (2) $q(C_1) \ge r$. *We denote*

$$\lambda(r) = \inf_{A \in \Psi(r)} M(\Gamma_A).$$

Theorem 2.5.2. *The function* $\lambda : (0,1] \to \mathbb{R}$ *has the following properties:*

(1) λ *is increasing.*

(2) $lim_{r\to 0}\lambda(r) = 0$.

(3) $\lambda(r) > 0$ *for every* $0 < r \le 1$.

Proof. (1) Suppose that $r < s$. We claim that $\Psi(r) \supset \Psi(s)$. Let $A = R(C_0, C_1) \in \Psi(s)$. Then $q(C_0) \ge s$ and $q(C_1) \ge s$. Since $r < s$, we have $q(C_0) \ge r$ and $q(C_1) \ge r$. Thus $A \in \Psi(r)$. Therefore $\lambda(r) \le \lambda(s)$. That is, λ is increasing.

(2) Let $0 < r < 1/2$ and $A = B(0, r) \smallsetminus \overline{B(0, r^2)} \in \Psi(r^2)$. By Example 3, we get

$$M(\Gamma_A) = \frac{2\pi}{\log \frac{r}{r^2}} = \frac{2\pi}{\log \frac{1}{r}}.$$

Thus $M(\Gamma_A) \to 0$ as $r \to 0$. Hence $lim_{r\to 0}\lambda(r) = 0$.

(3) We want to show that $\lambda(r) > 0$ for every $0 < r \le 1$. Let $0 < r \le 1$. Suppose that $A = R(C_0, C_1) \in \Psi(r)$. Choose $a, b \in C_0$ and $c, d \in C_1$ such that $q(a, b) \ge r$ and $q(c, d) \ge r$. Performing a spherical isometry of $\overline{\mathbb{R}^2}$ we may assume that $d = \infty$. By Theorem 2.3.6, we have

$$M(\Gamma_A) \ge \xi\left(\frac{|c-a|}{|b-a|}\right).$$

Next, we estimate $\frac{|c-a|}{|b-a|}$. Without loss of generality we assume that $|a| \le |b|$. We obtain

$$r \le q(a, b) = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} \le \frac{|a-b|}{1+|a|^2}$$

and

$$r \le q(c, \infty) = \frac{1}{\sqrt{1+|c|^2}} \le \frac{1}{|c|}.$$

Thus

$$\frac{r}{|a-b|} \le \frac{1}{1+|a|^2} \qquad \text{and} \qquad |c| \le \frac{1}{r}.$$

That is,

$$\frac{1}{|a-b|} \le \frac{1}{r(1+|a|^2)} \qquad \text{and} \qquad |c-a| \le |c| + |a| \le \frac{1+r|a|}{r}.$$

Hence

$$\frac{|c-a|}{|b-a|} \leq \frac{1+r|a|}{r^2(1+|a|^2)}.$$

Since $a \neq \infty$,

$$u(r) = \max_{0 \leq t < \infty} \frac{1+rt}{r^2(1+t^2)} < \infty.$$

Thus

$$\frac{|c-a|}{|b-a|} \leq u(r).$$

By Theorem 2.3.5(1), we get

$$\xi(u(r)) \leq \xi\left(\frac{|c-a|}{|b-a|}\right) \leq M(\Gamma_A).$$

Since $u(r) > 0$, Theorem 2.3.5(4) implies that $\xi(u(r)) > 0$. Thus $M(\Gamma_A) > 0$ for every $A \in \Psi(r)$. Hence $\lambda(r) > 0$.

□

Definition 2.5.3. *Given $0 < r \leq 1$ and $0 < t \leq 1$, let $\Psi(r,t)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^2}$ with the following properties: (1) $q(C_0) \geq r$, (2) $q(C_1) \geq r$, (3) $q(C_0, C_1) \leq t$. We denote*

$$\lambda(r,t) = \inf_{A \in \Psi(r,t)} M(\Gamma_A).$$

Note that $\lambda(r,1)$ is equal to the number $\lambda(r)$.

Theorem 2.5.4. *The function $\lambda : (0,1] \times (0,1] \to \mathbb{R}$ has the following properties:*

(1) $\lambda(r,t)$ *is increasing in* r.

(2) $\lambda(r,t)$ *is decreasing in* t.

(3) $\lambda(r,t) \geq \lambda(r) > 0$ *for every* r *and* t.

(4) $lim_{t\to 0}\lambda(r,t) = \infty$ *for every* r.

Proof. (1) and (2) are obvious.

(3) Since $\lambda(r,t)$ is decreasing in t and $\lambda(r) > 0$ for every $0 < r \leq 1$, we have $\lambda(r,t) \geq \lambda(r) > 0$ as desired.

(4) Let $0 < r \leq 1$. Suppose that $t < \frac{r}{4}$ and $A = R(C_0, C_1) \in \Psi(r,t)$. Pick $a \in C_0$ and $c \in C_1$ such that $q(a,c) \leq t$. Next choose $b \in C_0$ and $d \in C_1$ such that

$$q(a,b) \geq \frac{r}{2} \qquad \text{and} \qquad q(c,d) \geq \frac{r}{2}.$$

Performing a spherical isometry of $\overline{\mathbb{R}^2}$, we may assume that $d = \infty$. Hence

$$M(\Gamma_A) \geq \xi\left(\frac{|c-a|}{|b-a|}\right).$$

Since

$$\frac{r}{2} \leq q(c,\infty) = \frac{1}{\sqrt{1+|c|^2}} \leq \frac{1}{|c|}$$

and

$$\frac{r}{4} \leq \frac{r}{2} - t \leq q(c,\infty) - q(a,c) \leq q(a,\infty) \leq \frac{1}{|a|},$$

we obtain

$$|c-a| = q(a,c)\sqrt{1+|a|^2}\sqrt{1+|c|^2} \leq t\left(1+\frac{16}{r^2}\right).$$

Since

$$|b-a| \geq q(a,b) \geq \frac{r}{2},$$

we have

$$\frac{|c-a|}{|b-a|} \leq \frac{2t}{r}\left(1+\frac{16}{r^2}\right).$$

This implies that

$$\xi\left(\frac{2t}{r}\left(1+\frac{16}{r^2}\right)\right) \leq \xi\left(\frac{|c-a|}{|b-a|}\right) \leq M(\Gamma_A).$$

Hence

$$\lambda(r,t) \geq \xi\left(\frac{2t}{r}\left(1+\frac{16}{r^2}\right)\right) \qquad \text{for } t < \frac{r}{4}.$$

Since $\xi(s) \to \infty$ as $s \to 0$, this proves (4). $\square$

Chapter 3

Normal Families

This section is devoted to an investigation of Montel's theorem in the setting of conformal mappings. We begin by recalling the definition of equicontinuity.

3.1 Equicontinuity

Definition 3.1.1. *Suppose that T is a topological space, that (M,q) is a metric space and that $\mathscr{F}$ is a family of mappings $f : T \to M$. The family $\mathscr{F}$ is* ***equicontinuous at a point*** *$x_0 \in T$ if for each $\epsilon > 0$ there is a neighborhood U of x_0 such that*

$$q(f(x), f(x_0)) < \epsilon \text{ whenever } x \in U \text{ and } f \in \mathscr{F}.$$

If $\mathscr{F}$ is equicontinuous at each point of T, it is called ***equicontinuous****.*

In our case, T will always be a domain in $\overline{\mathbb{R}^2}$, $M = \overline{\mathbb{R}^2}$, and q is the chordal metric.

Example 5.

1. *If $\mathscr{F}$ consists of only finitely many elements of continuous functions, then $\mathscr{F}$ is equicontinuous.*

2. *Let $\mathscr{F}_1 = \{f_n : \mathbb{R}^2 \to \mathbb{R}^2 | f_n(z) = z + n, n \in \mathbb{N}\}$. Then $\mathscr{F}_1$ is equicontinuous everywhere.*

3. *Let $\mathscr{F}_2 = \{f_n : \mathbb{R}^2 \to \mathbb{R}^2 | f_n(z) = nz, n \in \mathbb{N}\}$. Then $\mathscr{F}_2$ is equicontinuous nowhere.*

The following theorem is our main result on equicontinuity:

Theorem 3.1.2. *Let $\mathscr{F}$ be a family of conformal mappings of a domain D into $\overline{\mathbb{R}^2}$. If each $f \in \mathscr{F}$ omits 2 values a_f, b_f with chordal distance*

$$q(a_f, b_f) \geq r,$$

where $r > 0$ is fixed, then $\mathscr{F}$ is equicontinuous.

Proof. Let $x_0 \in D$ and $0 < \epsilon < r$. Choose neighborhoods U and V of x_0, for example disks, so that $\overline{U} \subset V \subset D$, that $A = V \setminus \overline{U}$ is a ring domain, and so that $M(\Gamma_A) < \lambda(\epsilon)$, where λ is the function introduced in Definition 2.5.1. Let $f \in \mathscr{F}$. Then $f(A) = R(C_0, C_1)$, where $C_0 = f(\overline{U})$ and $C_1 = (f(V))^c$, is a ring. Since C_1 contains a_f and b_f, we get

$$q(C_1) \geq q(a_f, b_f) \geq r.$$

For each $x \in U$, the point $f(x)$ lies in C_0. Thus

$$q\left(f(x), f(x_0)\right) \leq q(C_0).$$

Let $t = min\left(r, q\left(f(x), f(x_0)\right)\right)$. Then $q(C_0) \geq t$, $q(C_1) \geq t$, and $f(A) \in \Psi(t)$, where $\Psi(t)$ is the collection of rings introduced in Definition 2.5.1. By the definition of $\lambda(t)$, we obtain

$$M(\Gamma_{f(A)}) \geq \lambda(t).$$

Since f is a conformal mapping, Theorem 2.2.6 implies

$$M(\Gamma_{f(A)}) = M(\Gamma_A) < \lambda(\epsilon).$$

Thus $\lambda(t) < \lambda(\epsilon)$. Since λ is increasing and $t = min\left(r, q\left(f(x), f(x_0)\right)\right)$, we have $t < \epsilon < r$ and $t = q\left(f(x), f(x_0)\right)$. Hence

$$q\left(f(x), f(x_0)\right) < \epsilon.$$

This holds for all $x \in U$ and $f \in \mathscr{F}$. Therefore $q\left(f(x), f(x_0)\right) < \epsilon$ whenever $x \in U$ and $f \in \mathscr{F}$. Consequently, $\mathscr{F}$ is equicontinuous at x_0.

□

Theorem 3.1.3. *Let $\mathscr{F}$ be a family of conformal mappings of a domain D. Then $\mathscr{F}$ is equicontinuous if one of the following conditions is satisfied:*

(1) *There are points $x_1, x_2 \in D$ and a number $r > 0$ such that each $f \in \mathscr{F}$ omits a point a_f and $q\left(a_f, f(x_i)\right) \geq r$ for $i = 1, 2$.*

(2) *There are points $x_1, x_2, x_3 \in D$ and a number $r > 0$ such that each $f \in \mathscr{F}$ satisfies the three inequalities $q\left(f(x_i), f(x_j)\right) \geq r$, for $i \neq j$.*

Proof. (1) Set $D_1 = D \setminus \{x_1\}$. Then every $f|D_1$ omits the points a_f and $f(x_1)$ such that $q(a_f, f(x_1)) \geq r$. By Theorem 3.1.2, the family of all restrictions $f|D_1$ is equicontinuous. That is, $\mathscr{F}$ is equicontinuous at every point of D except possibly at x_1. Considering similarly the restrictions $f|D \setminus \{x_2\}$, we conclude that $\mathscr{F}$ is equicontinuous also at x_1. Hence $\mathscr{F}$ is equicontinuous in D.

(2) Every $f|D \setminus \{x_1, x_2\}$ omits the points $f(x_1)$ and $f(x_2)$. By Theorem 3.1.2, the family $\mathscr{F}$ is equicontinuous at every point of D except possibly at x_1 and x_2. Considering similarly the restrictions $f|D \setminus \{x_2, x_3\}$ and $f|D \setminus \{x_1, x_3\}$, we conclude that $\mathscr{F}$ is equicontinuous also at these points.

□

Corollary 3.1.4. *If $\mathscr{F}$ be a family of conformal mappings of a domain D such that each $f \in \mathscr{F}$ assume at three given points three fixed values, then $\mathscr{F}$ is equicontinuous.*

3.2 Normal Family

Definition 3.2.1. *Suppose that T is a topological space, that (M, q) is a metric space and that (f_n) is a sequence of mappings from T into M.*

1. *(f_n) **converges** in T **pointwise** to a mapping f if for each $z \in T$*

$$f_n(z) \to f(z) \text{ as } n \to \infty.$$

2. *(f_n) **converges** in T **uniformly** to a mapping f if*

$$\sup_{z \in T} |f_n(z) - f(z)| \to 0 \text{ as } n \to \infty.$$

3. *(f_n) **converges** in T **c-uniformly** to a mapping f if $f_n \to f$ uniformly on every compact subset of T.*

Definition 3.2.2. *Let T be a topological space and let (M, q) be a metric space. A family $\mathscr{F}$ of continuous mappings $f : T \to M$ is called a **normal family** if every sequence in $\mathscr{F}$ has a subsequence that converges c-uniformly in T.*

Remark 3.2.3. *If $\mathscr{F}$ is a normal family, then every sequence in $\mathscr{F}$ contains a subsequence that converges pointwise in all of T.*

Theorem 3.2.4. *Let (M, q) be complete, and let $f_n : T \to M$ be an equicontinuous sequence that converges at every point of a set E which is dense in T. Then (f_n) converges c-uniformly in T.*

Proof. See [4, Lemma 4.2, page 281]. □

The link between equicontinuity and normality is the following celebrated theorem:

Theorem 3.2.5. ***(Ascoli's theorem)*** *If T is a separable topological space and M is a compact metric space, then every equicontinuous family $\mathscr{F}$ of mappings $f : T \to M$ is a normal family.*

Proof. See [5, Theorem 20.4, page 68]. □

Combining Ascoli's theorem with Theorem 3.1.2 we obtain the following version of Montel's theorem:

Theorem 3.2.6. *Let $\mathscr{F}$ be a family of conformal mappings of a domain D into $\overline{\mathbb{R}^2}$. If each $f \in \mathscr{F}$ omits 2 values a_f, b_f with chordal distance*

$$q(a_f, b_f) \geq r,$$

where $r > 0$ is fixed, then $\mathscr{F}$ is a normal family.

Chapter 4

Convergence

In this section we investigate the limit mapping of a convergent sequence of conformal mappings. In particular, we are interested in determining what kind of limit mappings are possible. It turns out that there are 3 possibilities. To achieve this we need three auxiliary results.

4.1 Convergence of conformal mappings

Lemma 4.1.1. *Let (f_n) be a sequence of analytic functions in an open subset D of $\mathbb{R}^2$. If $f_n \to f$ uniformly on every compact subset of D, then f is analytic in D.*

Proof. Let $z \in D$, let U be a neighborhood of z and let T be a triangle in U. Then T is a compact subset of D. By Cauchy's theorem, we get $\int_T f_n = 0$ for all n. Since $f_n \to f$ uniformly on T,

$$\int_T f = \int_T \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_T f_n = 0.$$

By Morera's theorem, f is analytic in U. This holds for every z in D, and so f is analytic in D.

□

In the following, we denote D_n the image of D under f for each n.

Lemma 4.1.2. *Let $f_n : D \to D_n$ be conformal, let $f_n \to f$ pointwise in D, and let $\{f_n | n \in \mathbb{N}\}$ be equicontinuous in D. If there are distinct points $z_1, z_2 \in D$ such that $f(z_1) = f(z_2)$, then every neighborhood U of z_1 contains a point $x_0 \neq z_1$ such that $f(z_1) = f(x_0)$.*

Proof. Let U be a neighborhood of z_1 not containing z_2. We choose a circle $S \subset U$ such that S separates the points z_1, z_2 in $\overline{\mathbb{R}^2}$. Since f_n is a conformal mapping, hence a homeomorphism, the set $f_n(S)$ separates $f_n(z_1)$ and $f_n(z_2)$ for all n. Thus there are points $x_n \in S$ such that

$$q\left(f_n(x_n), f_n(z_1)\right) \leq q\left(f_n(z_2), f_n(z_1)\right).$$

As a compact set, S contains all its limit points. Let x_0 be a limit point. Thus we may assume that $x_n \to x_0 \in S$. Next, we will show that

$$q\left(f_n(x_n), f(x_0)\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\epsilon > 0$. Since $\{f_n | n \in \mathbb{N}\}$ is equicontinuous at x_0, there is a neighborhood V of x_0 such that $q\left(f_n(x), f_n(x_0)\right) < \frac{\epsilon}{2}$ whenever $x \in V$ and $n \in \mathbb{N}$. Thus there is $M_0 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq M_0$, and such that

$$q\left(f_n(x_n), f_n(x_0)\right) < \frac{\epsilon}{2}$$

whenever $n \geq M_0$. Since $f_n \to f$ pointwise in D, there is $M_1 \in \mathbb{N}$ such that

$$q\left(f_n(x_0), f(x_0)\right) < \frac{\epsilon}{2}$$

for all $n \geq M_1$. Let $M = \max\{M_0, M_1\}$. Then

$$\begin{aligned} q\left(f_n(x_n), f(x_0)\right) &\leq q\left(f_n(x_n), f_n(x_0)\right) + q\left(f_n(x_0), f(x_0)\right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $n \geq M$. Since ϵ is arbitrary, we have

$$q\left(f_n(x_n), f(x_0)\right) \to 0 \quad \text{as} \quad n \to \infty$$

as desired. Now

$$\begin{aligned} q\left(f_n(x_0), f_n(z_1)\right) &\leq q\left(f_n(x_0), f(x_0)\right) + q\left(f(x_0), f_n(x_n)\right) + q\left(f_n(x_n), f_n(z_1)\right) \\ &\leq q\left(f_n(x_0), f(x_0)\right) + q\left(f(x_0), f_n(x_n)\right) + q\left(f_n(z_2), f_n(z_1)\right) \end{aligned}$$

for all n. By taking the limits as $n \to \infty$, obtain

$$q\left(f(x_0), f(z_1)\right) \leq q\left(f(z_2), f(z_1)\right).$$

By the hypothesis, $f(z_1) = f(z_2)$, and we obtain $q\left(f(x_0), f(z_1)\right) = 0$, which means that $f(z_1) = f(x_0)$. Hence the proof is complete.

□

Lemma 4.1.3. *Let $f_n : D \to D_n$ be conformal, let $f_n \to f$ pointwise in D, and let $\{f_n | n \in \mathbb{N}\}$ be equicontinuous in D. Then every $x_0 \in D$ has a neighborhood U such that $f|U$ is either injective or constant.*

Proof. Since $\{f_n | n \in \mathbb{N}\}$ is equicontinuous, there is a ball neighborhood U of x_0 such that

$$q\left(f_n(x), f_n(x_0)\right) \leq \frac{1}{4} \qquad \text{whenever } x \in U \text{ and } n \in \mathbb{N}.$$

If U does not have the desired property, we can pick distinct points u_1, u_2, u_3 in U such that $f(u_1) \neq f(u_2) = f(u_3)$. We join u_1 and u_2 by an arc $J_0 \subset U$ and choose another arc J_1 such that the end points of J_1 are u_3 and a point $u_4 \in \partial U$ and such that $J_1 \setminus U = u_4$, $J_0 \cap J_1 = \phi$. If A is the ring $U \setminus (J_0 \cup J_1)$, then $f_n(A) = A_n$ is the ring $R(C_0^n, C_1^n)$ where $C_0^n = f_n(J_0)$ and $C_1^n = (f_n(U \setminus J_1))^c$. Since $f_n(u_1)$ and $f_n(u_2)$ lie in C_0^n, we have

$$q(C_0^n) \geq q\left(f_n(u_1), f_n(u_2)\right).$$

Because $q\left(f_n(U)\right) \leq \frac{1}{2}$, it follows that $f_n(U)$ under stereographic projection must be contained in a half-sphere, so its complement contains a half-sphere, and therefore $q\left((f_n(U))^c\right) = 1$. Thus

$$1 = q\left((f_n(U))^c\right) \leq q\left(C_1^n\right).$$

Now $f_n(u_2) \in C_0^n$ and $f_n(u_3) \in C_1^n$, and so

$$q\left(C_0^n, C_1^n\right) \leq q\left(f_n(u_2), f_n(u_3)\right).$$

The ring A_n, therefore, lies in the collection $\Psi(r_n, t_n)$, where

$$r_n = q\left(f_n(u_1), f_n(u_2)\right) \quad \text{and} \quad t_n = q\left(f_n(u_2), f_n(u_3)\right).$$

Recalling that

$$\lambda(r_n, t_n) = \inf_{B \in \Psi(r_n, t_n)} M(\Gamma_B),$$

we thus obtain

$$M(\Gamma_{A_n}) \geq \lambda(r_n, t_n)$$

for all $n \in \mathbb{N}$. Now, since $f_n \to f$ pointwise in D, it follows that

$$r_n \to q\left(f(u_1), f(u_2)\right) > 0$$

and

$$t_n \to q\left(f(u_2), f(u_3)\right) = 0.$$

By Theorem 2.5.4(4), therefore,

$$\lambda(r_n, t_n) \to \infty$$

as $n \to \infty$. Hence $M(\Gamma_{A_n}) \to \infty$. Since f_n is conformal for each n,

$$M(\Gamma_{A_n}) = M(\Gamma_A),$$

by virtue of Theorem 2.2.6. Thus $M(\Gamma_A) = \infty$. But since A is a ring, $M(\Gamma_A)$ is finite by Theorem 2.3.2. Thus we have reached a contradiction. Hence every $x_0 \in D$ has a neighborhood U such that $f|U$ is either injective or constant.

□

The main result of this section is the following theorem.

Theorem 4.1.4. *If $f_n : D \to D_n$ is conformal and $f_n \to f$ pointwise in D, then there are 3 possibilities:*

(1) *f is a constant. The convergence may be c-uniform or not.*

(2) *f assumes exactly 2 values, one of which at exactly one point. The convergence is not c-uniform.*

(3) *f is a conformal mapping. The convergence is c-uniform.*

Proof. Suppose first that f assumes exactly 2 values, say $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Since $f_n \to f$ pointwise, we get

$$f_n(a_1) \to f(a_1) = b_1 \text{ and } f_n(a_2) \to f(a_2) = b_2.$$

Let V_1 and V_2 be neighborhoods of b_1 and b_2, respectively, such that $\overline{V_1} \cap \overline{V_2} = \phi$. Then there exists an integer n_0 such that $f_n(a_1) \in V_1$ and $f_n(a_2) \in V_2$ for all $n > n_0$. For $n = 1, 2, \ldots, n_0$, we have $f_n(a_1) \neq f_n(a_2)$ because f_n is injective. Let

$$r = \min \left\{ q(f_1(a_1), f_1(a_2)), q(f_2(a_1), f_2(a_2)), \ldots, q(f_{n_0}(a_1), f_{n_0}(a_2)), q(\overline{V_1}, \overline{V_2}) \right\}.$$

Then $r > 0$. Thus $f_n|D \setminus \{a_1, a_2\}$ omits 2 values, which are $f_n(a_1)$ and $f_n(a_2)$ with

$$q(f_n(a_1), f_n(a_2)) \geq r > 0$$

for all $n \in \mathbb{N}$. By Theorem 3.1.2, the family $\{f_n|D \setminus \{a_1, a_2\} \,|n \in \mathbb{N}\}$ is equicontinuous. Thus $f_n \to f$ c-uniformly in $D \setminus \{a_1, a_2\}$. This implies that f is continuous in $D \setminus \{a_1, a_2\}$. Since $f(D) = \{b_1, b_2\}$, it must be the case that $f(D \setminus \{a_1, a_2\})$ is either $\{b_1\}$ or $\{b_2\}$. Suppose that $f(D \setminus \{a_1, a_2\}) = \{b_1\}$. Thus we have

$$f(x) = \begin{cases} b_1 & \text{for } x \in D \setminus \{a_2\}, \\ b_2 & \text{for } x = a_2. \end{cases}$$

Here, f is not continuous in D. Thus the convergence cannot be c-uniform and we have the situation (2).

It remains to prove that if f assumes at least 3 values, say $b_1 = f(a_1)$, $b_2 = f(a_2)$, and $b_3 = f(a_3)$, then we have the situation (3). Since $f_n \to f$ pointwise in D, we get

$$f_n(a_1) \to f(a_1) = b_1, f_n(a_2) \to f(a_2) = b_2, \text{ and } f_n(a_3) \to f(a_3) = b_3.$$

Let V_1, V_2 and V_3 be neighborhoods of b_1, b_2, and b_3, respectively, such that $\overline{V_i} \cap \overline{V_j} = \phi$ for all $i \neq j \in \{1, 2, 3\}$. Then there exists an integer n_1 such that $f_n(a_1) \in V_1, f_n(a_2) \in V_2$ and $f_n(a_3) \in V_3$ for all $n > n_1$. For $n = 1, 2, \ldots, n_1$, we have $f_n(a_i) \neq f_n(a_j)$ for all $i \neq j$. Let

$$r = \min \left\{ q(f_1(a_i), f_1(a_j)), q(f_2(a_i), f_2(a_j)), \ldots, q(f_{n_1}(a_i), f_{n_1}(a_j)), q(\overline{V_i}, \overline{V_j}) \right\}$$

for all $i \neq j \in \{1, 2, 3\}$. Then $r > 0$. Thus

$$q(f_n(a_i), f_n(a_j)) \geq r > 0$$

for all $n \in \mathbb{N}$. By Theorem 3.1.3, $\{f_n | n \in \mathbb{N}\}$ is equicontinuous. By Theorem 3.2.4, $f_n \to f$ c-uniformly. Hence f is continuous in D.

Next, we want to show that f is injective. Suppose that there are distinct points $z_1, z_2 \in D$ such that $f(z_1) = f(z_2)$. By Lemma 4.1.2, every neighborhood U of z_1 contains a point $x_0 \neq z_1$ such that $f(z_1) = f(x_0)$. By Lemma 4.1.3, every $z \in D$ has a neighborhood U such that $f|U$ is either injective or constant. Let D_1 be the set of all $x \in D$ which have a neighborhood in which f is injective, and D_2 the set of all $x \in D$ which have a neighborhood in which f is constant. Then D_1 and D_2 are open and disjoint, and $D = D_1 \cup D_2$. Since $z_1 \notin D_1$, we have $D_2 \neq \phi$. Since D is connected, $D = D_2$. Hence f is constant in D, which is a contradiction. Therefore f is injective.

Since f_n is conformal and $f_n \to f$ c-uniformly, f is analytic in D by Lemma 4.1.1. As an analytic function, f is an open mapping. Hence $f(D)$ is an open set. As a connected set, therefore, $f(D)$ is a domain. Since f is injective, we can infer that f is a continuous bijection of D onto $f(D)$. The openness of f guarantees that also f^{-1} is continuous. Therefore $f : D \to f(D)$ is an analytic homeomorphism. Such a mapping is always a conformal mapping.

□

Corollary 4.1.5. *If $f_n : D \to D_n$ is conformal and $f_n \to f$ c-uniformly in D, then f is either a conformal mapping onto a domain D' or a constant.*

If D happens to be the extended plane $\overline{\mathbb{R}^2}$, the second case above can be ruled out:

Theorem 4.1.6. *If $f_n : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is conformal and $f_n \to f$ c-uniformly in $\overline{\mathbb{R}^2}$, then f is a conformal mapping onto $\overline{\mathbb{R}^2}$.*

Proof. By Corollary 4.1.5, f is either a conformal mapping onto a domain in $\overline{\mathbb{R}^2}$ or a constant. If f is a constant, then $q\left(f_n(\overline{\mathbb{R}^2})\right) \to 0$. Since $f_n(\overline{\mathbb{R}^2}) = \overline{\mathbb{R}^2}$, this is impossible. Hence f must be a conformal mapping onto $\overline{\mathbb{R}^2}$.

□

Remark 4.1.7. *In Theorem 4.1.6, it is possible that (f_n) converges non c-uniformly to a constant. For example: the sequence (f_n), where $f_n(z) = z + ne_1$ for all $z \in \overline{\mathbb{R}}^2$ and $e_1 = (1, 0)$, converges to a constant.*

Proof. Let $z \in \overline{\mathbb{R}^2}$. Since

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} (z + ne_1) = \infty,$$

(f_n) converges pointwise to the constant function f defined by $f(z) = \infty$ for all $z \in \overline{\mathbb{R}^2}$. Next, we will show that (f_n) does not converge uniformly to f. Suppose that there is a positive integer n_0 such that for all $z \in \overline{\mathbb{R}^2}$,

$$q\left(f_n(z), f(z)\right) < \frac{1}{2}$$

whenever $n \geq n_0$. Let $z = -ne_1$. Then $f_n(z) = f_n(-ne_1) = (-ne_1) + ne_1 = 0$. But

$$q(f_n(z), f(z)) = q(0, \infty) = 1,$$

for all n. Thus $q(f_n(z), f(z))$ is not less than $\frac{1}{2}$ for all $n > n_0$, a contradiction. Therefore (f_n) does not converge c-uniformly to f.

□

Remark 4.1.8. *In Theorem 4.1.6, it is possible that (f_n) converges c-uniformly to a constant. For example, $f_n(z) = \frac{z}{n}$ for all $z \in \mathbb{R}^2_+$ where $\mathbb{R}^2_+$ is the upper half-plane.*

Proof. Let $z \in \mathbb{R}^2_+$. Note that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{z}{n} = 0.$$

Thus $f_n \to 0$ pointwise in $\mathbb{R}^2_+$. Next, we will show that $f_n \to 0$ c-uniformly in $\mathbb{R}^2_+$. Let $\epsilon > 0$ and F be a compact subset of $\mathbb{R}^2_+$. Let $M > 0$ such that $|z| \leq M$ for all $z \in F$. Choose a positive integer $N > \frac{M}{\epsilon}$. Then, for any $z \in F$,

$$|f_n(z) - 0| = \left|\frac{z}{n}\right| \leq \left|\frac{M}{N}\right| < \epsilon$$

whenever $n \geq N$. Hence $f_n \to 0$ uniformly in F.

□

To deal with the case where D has exactly one boundary point requires a consideration on the removability of an isolated boundary point. For this purpose, we recall the concept of a cluster set:

Definition 4.1.9. *Given a mapping $f : D \to \overline{\mathbb{R}^2}$ and a point $b \in \partial D$, the **cluster set** $C(f, b)$ of f at b is the set of all points $b' \in \overline{\mathbb{R}^2}$ such that there exists a sequence (x_j) such that $x_j \in D$, $x_j \to b$ and $f(x_j) \to b'$. Alternatively,*

$$C(f, b) = \bigcap \overline{f(D \cap U)}$$

where U runs through all neighborhoods of b. Thus f has a limit b' at b if and only if $C(f, b) = \{b'\}$. Since $\overline{\mathbb{R}^2}$ is compact, the cluster set is never empty. The cluster set of f on a set $A \subset \partial D$ is defined by

$$C(f, A) = \bigcup_{b\in A} C(f, b).$$

The cluster set of a homeomorphism $f : D \to D'$ is always a subset of $\partial D'$.

Suppose that D is a proper subdomain of $\overline{\mathbb{R}^2}$. Let C be a component of D^c. Then $\overline{D} \cap C$ is a component of ∂D. Thus if $f : D \to D'$ is a homeomorphism and B is a component of ∂D, then $C(f, B)$ is a component of $\partial D'$. That is, D and D' have the same number of boundary components. This implies that a homeomorphic image of a ring is always a ring.

Theorem 4.1.10. *Suppose that $f : D \to D'$ is a conformal mapping and that b is an isolated point of ∂D. Then f has a limit b' at b, and b' is an isolated point of $\partial D'$. Defining $f^*(b) = b'$ and $f^*|D = f$, we obtain a conformal mapping $f^* : D \cup \{b\} \to D' \cup \{b'\}$.*

Proof. Since b is an isolated point of ∂D, we can choose a ball neighborhood U of b such that $\overline{U} \cap \partial D = \{b\}$. Let $A = U \setminus \{b\}$. Then A is a ring with boundary components ∂U and $\{b\}$. Thus $M(\Gamma_A) = 0$ by Example 4. Moreover, $f(A)$ is also a ring with boundary components $f(\partial U)$ and $C(f, b)$. Since f is conformal, $M(\Gamma_f(A)) = M(\Gamma_A) = 0$ by Theorem 2.2.6. By Theorem 2.3.7, $C(f, b)$ consists of a single point b'. Thus f has a limit b' at b. Because f is a conformal mapping and $\{b\}$ is a component of ∂A, $C(f, b) = \{b'\}$ is a component of $\partial f(A)$. This implies that $q(b', f(\partial U)) > 0$. Since

$$q(b', \partial D' \setminus \{b'\}) \geq q(b', f(\partial U)) > 0,$$

b' is an isolated point of $\partial D'$. Note that $D \cup \{b\}$ is a domain. Then $f^* : D \cup \{b\} \to D' \cup \{b'\}$ is a continuous bijection, and hence a homeomorphism. That is, f^* is a conformal mapping.

□

Theorem 4.1.11. *Let $D = \overline{\mathbb{R}^2} \setminus \{a\}$ and let (f_n) be a sequence of conformal mappings of D that converges c-uniformly to a mapping f. Then f is either a constant or a conformal mapping onto a domain $\overline{\mathbb{R}^2} \setminus \{b\}$. In the second case, $b = \lim_{n\to\infty} b_n$ where $f_n(D) = \overline{\mathbb{R}^2} \setminus \{b_n\}$.*

Proof. Suppose that f is not constant. Then, by Corollary 4.1.5, f is conformal. For each $n \in \mathbb{N}$, since f_n is a conformal mapping, we have f_n is analytic on $D \setminus \{a\}$. By Theorem 4.1.10, f_n can be extended to a conformal mapping $f_n^* : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$. Set $f_n^*(a) = b_n$. Let x_1, x_2, x_3 be distinct points in D. Let

$$\epsilon = \min\{q(f(x_i), f(x_j)) : i \neq j\} > 0.$$

Since $f_n \to f$ c-uniformly in D, there is an integer N such that

$$q(f_n(x_j), f(x_j)) < \frac{\epsilon}{3}$$

for all $n \geq N$ and $j = 1, 2, 3$. Thus $q(f_n(x_i), f_n(x_j)) > \frac{\epsilon}{3}$ for all $n \geq N$ and $i \neq j$. Let

$$r_n = \min\{q(f_n(x_i), f_n(x_j)) : i \neq j\} \qquad \text{and} \qquad r_0 = \min\{r_n : n = 1, \ldots, N\}.$$

Set $r = \min\left\{\frac{r_0}{2}, \frac{\epsilon}{3}\right\}$. Then

$$q(f_n(x_i), f_n(x_j)) > r$$

whenever $n \in \mathbb{N}$ and $i \neq j$. By Theorem 3.1.3, $\{f_n^* | n \in \mathbb{N}\}$ is equicontinuous in $\overline{\mathbb{R}^2}$. Then (f_n^*) converges c-uniformly to a mapping f^* by Theorem 3.2.4. Thus f^* is a conformal mapping onto $\overline{\mathbb{R}^2}$ by Theorem 4.1.6 and $f_n^* : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ converges uniformly to f^*. Hence f maps D onto $\overline{\mathbb{R}^2} \setminus \{b\}$, where $b = f^*(a) = \lim_{n\to\infty} f_n^*(a) = \lim_{n\to\infty} b_n$.

□

The discussion of the case where D has more than one boundary point involves the concept of the kernel, next to be introduced:

Definition 4.1.12. *Let $E_1, E_2, \ldots$ be a sequence of sets in $\overline{\mathbb{R}^2}$. The **kernel** $\ker_{n\to\infty} E_n$ of this sequence is the set of all points in $\overline{\mathbb{R}^2}$ which have a neighborhood contained in all but a finite number of the set E_n. Equivalently,*

$$\ker_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{n=k}^{\infty} E_n,$$

where int E *is the interior of a set E.*

The kernel of a sequence is always an open set. However, it does not need to be connected ever if the sets E_n are domains. We shall also use the simpler but less rigorous notation $\ker E_n$ for $\ker_{n\to\infty} E_n$.

Theorem 4.1.13. *Let D be a domain which has at least two boundary points. Let $f_n : D \to D_n$ be a sequence of conformal mappings that converges c-uniformly to a mapping f. Then f is either a conformal mapping onto a component of* $\ker D_n$ *or a constant in* $(\ker D_n \cup \ker D_n^c)^c$.

Proof. Suppose that f is a conformal mapping onto D'. We first show that $D' \subset \ker D_n$. Let $y_0 \in D'$. We will prove that $y_0 \in \ker D_n$. That is, we will show that there is a neighborhood U of y_0 and an integer n_0 such that $U \subset D_n$ for all $n \geq n_0$.

Let $x_0 = f^{-1}(y_0)$. Choose a neighborhood U of x_0 such that $\overline{U} \subset D$. Since $f_n \to f$ c-uniformly in D, $f_n \to f$ uniformly in $\overline{U}$. Thus $f_n \to f$ pointwise in $\overline{U}$ which implies that $f_n(x_0) \to f(x_0) = y_0$. Then there is a ball neighborhood V of y_0 and an integer n_0 such that

$$f_n(x_0) \in V \qquad \text{and} \qquad V \cap f_n(\partial U) = \phi$$

for all $n \geq n_0$. Since V is conected,

$$V \subset f_n(U) \subset D_n$$

for all $n \geq n_0$. Thus $y_0 \in \ker D_n$. Hence $D' \subset \ker D_n$.

Let G be the component of $\ker D_n$ which contains D'. Thus $D' \subset G$. Suppose that $D' \neq G$. Then there is a point $b \in G \cap \partial D'$. By the definition of the kernel, there is a ball neighborhood U of b and an integer n_0 such that $U \subset D_n$ for $n \geq n_0$. Hence $g_n = f_n^{-1}|U$ is defined for $n \geq n_0$. Since every g_n omits two fixed values, namely the boundary points of D, $\{g_n | n \geq n_0\}$ is a normal family by Theorem 3.2.6. Passing to a subsequence, we may assume that $g_n \to g$ c-uniformly in U. Consider a point $x \in f^{-1}(D' \cap U)$. Since $f_n(x) \to f(x)$ and $\{g_n | n \geq n_0\}$ is equicontinuous at $f(x)$,

$$q(g_n(f(x)), x) = q(g_n(f(x)), g_n(f_n(x))) \to 0.$$

Thus $g(f(x)) = x$ for all $x \in f^{-1}(D' \cap U)$. In particular, g is not constant in the non-empty open set $D' \cap U$. By Corollary 4.1.5, g is a conformal mapping of U onto a domain V. From the first part it follows that

$$V \subset \ker g_n(U) \subset D.$$

Thus $g(b) \in D$. Since $f(g(y)) = y$ for all $y \in D' \cap U$, we have

$$f(g(b)) = \lim_{i \to \infty} f(g(y_i)) = \lim_{i \to \infty} y_i = b,$$

where $y_1, y_2, \ldots$ is any sequence in $D' \cap U$ converging to b. Thus $b \in D'$, a contradiction. Hence $D' = G$.

If f is not a conformal mapping, then by Corollary 4.1.5, $f(x) = c$ where c is a constant. Since every neighborhood U of c meets D_n for large n, there exists an integer n_0 such that $U \not\subseteq D_n^c$ for $n \geq n_0$. Therefore $c \in (\ker D_n^c)^c$. Next, we will show that $c \notin \ker D_n$. Assume that $c \in \ker D_n$. By the definition of $\ker D_n$, we can choose a ball neighborhood U of c and an integer n_0 such that $U \subset D_n$ for $n \geq n_0$. Define $g_n = f_n^{-1}|U$. Since every g_n omits two fixed values, $\{g_n | n \geq n_0\}$ is a normal family by Theorem 3.2.6. Passing to a convergent subsequence, we may assume that $g_n \to g$ c-uniformly in U. Let $x \in D$. Then $f_n(x) \in U$ for large n. Since $f_n(x) \to f(x)$ and $\{g_n | n \geq n_0\}$ is equicontinuous at $f(x)$,

$$q(x, g_n(c)) = q(g_n(f_n(x)), g_n(c)) \to 0.$$

Hence $(g_n(c))$ converges to every point $x \in D$, which gives a contradiction. Therefore $c \notin \ker D_n$. Now we conclude that $c \in (\ker D_n \cup \ker D_n^c)^c$.

□

Next we will show that if f is a conformal mapping, then the inverse mappings f_n^{-1} converge to f^{-1}.

Theorem 4.1.14. *Suppose that $f_n : D \to D_n$ is a sequence of conformal mappings that converges to a conformal mapping $f : D \to D'$. Then for every compact set $F \subset D'$ there is a integer n_0 such that $F \subset D_n$ for $n \geq n_0$. Moreover, the mappings $f_n^{-1}|F$ converges uniformly to $f^{-1}|F$.*

Proof. Suppose first that $F \neq D'$. This is the case if $D' \neq \overline{\mathbb{R}^2}$. We can choose a domain G such that $F \subset G$ and $\overline{G}$ is a proper subset of D'. By Theorem 4.1.13, we have $D' \subset \ker D_n$. Thus, for every $y \in \overline{G}$, we can find a neighborhood $U(y)$ of y and an integer $n(y)$ such that $U(y) \subset D_n$ for all $n \geq n(y)$. Choose a finite covering $\{U(y_1), \ldots, U(y_k)\}$ of $\overline{G}$, and set $n_0 = \max\{n(y_1), \ldots, n(y_k)\}$. Then $F \subset \overline{G} \subset D_n$ for $n \geq n_0$ as desired.

For $n \geq n_0$, let $g_n = f_n^{-1}|G$. If a_1, a_2 are two points in $D \backslash f^{-1}(\overline{G})$, $f_n(a_i) \notin \overline{G}$ for large n. Theorem 3.2.6 implies that $\{g_n | n \geq n_0\}$ is a normal family. By Theorem 3.2.4, it suffices to prove that $g_n(y_0) \to f^{-1}(y_0)$ for an arbitrary $y_0 \in G$. Fix y_0, let $\epsilon > 0$ and set $x_0 = f^{-1}(y_0)$. By the equicontinuity of the family $\{g_n\}$, choose a neighborhood $U \subset G$ of y_0 such that

$$q\left(g_n(y), g_n(y_0)\right) < \epsilon$$

for all $y \in U$ and $n \geq n_0$. Since $f_n(x_0) \to f(x_0) = y_0$, there is an integer $n_1 \geq n_0$ such that $f_n(x_0) \in U$ for $n \geq n_1$. Thus, for $n \geq n_1$, we have

$$q\left(x_0, g_n(y_0)\right) = q\left(g_n\left(f_n(x_0)\right), g_n(y_0)\right) < \epsilon.$$

Therefore $g_n(y_0) \to x_0$. This implies that $f_n^{-1}(y_0) \to f^{-1}(y_0)$. By Theorem 3.2.4, $(f_n^{-1}|F)$ converges uniformly to $f^{-1}|F$. Thus $g_n \to f^{-1}$ c-uniformly in G.

Finally, if $F = D' = \overline{\mathbb{R}^2}$, then we can choose compact proper subsets $F_1, F_2 \subset D'$ such that $F_1 \cup F_2 = F$. By the previous argument, we get that $f_n^{-1} \to f^{-1}$ uniformly in F_1 and F_2, respectively. Hence $f_n^{-1} \to f^{-1}$ uniformly in F.

□

Theorem 4.1.15. *Suppose that D is a domain which has at least two boundary points and that (f_n) is a sequence of conformal mappings onto a fixed domain D' such that $f_n \to f$ pointwise in D. Then D' has at least two boundary points, and the convergence is c-uniform. The limit mapping f is either a conformal mapping onto D' or a constant $c \in \partial D'$. In the first case, $f_n^{-1} \to f^{-1}$ c-uniformly in D'. The second case can occur only in the following cases:* (1) *∂D is connected.* (2) *∂D consists of two points.* (3) *∂D has an infinite number of components.*

Proof. Applying Theorem 4.1.10 to f_n^{-1}, we obtain that $\partial D'$ has at least two points. Thus f omits two fixed values. Theorem 3.2.6 implies that $\{f_n | n \in \mathbb{N}\}$ is a normal family. Thus $f_n \to f$ c-uniformly. Then f is either a conformal mapping onto D' or a constant $c \in \partial D'$ by Theorem 4.1.13.

Suppose that f is a conformal mapping. From Theorem 4.1.14, it follows that $f_n^{-1} \to f^{-1}$ c-uniformly in D'. Thus we have the first case.

Next, assume that f is a constant and ∂D has exactly k components $B_1, \ldots, B_k$, where $2 \leq k < \infty$. We will show that $k = 2$. Since f_n is a conformal mapping, $\partial D'$ has exactly k components $B'_1, \ldots, B'_k$ such that, for each n, B'_i is one of the cluster sets $C(f_n, B_m)$. Passing to a subsequence, we may assume that $B'_i = C(f_n, B_i)$ for all $n \in \mathbb{N}$ and $1 \leq i \leq k$. Choose a compact set $F \subset D$ such that the sets B_i are contained in different components of F^c. For example, we may put

$$F = \{x \in D | q(x, \partial D) \geq r\}$$

for a sufficiently small r. The sets B_i' are contained in different component of $(f_n(F))^c$. Since $f_n(x) \to c$ uniformly in F,

$$q\left(f_n(F)\right) \to 0.$$

Let U_n be the component of $(f_n(F))^c$ which has the largest spherical diameter. Note that

$$U_n \subset (f_n(F))^c \qquad \text{and} \qquad \partial(U_n^c) = \partial U_n \subset f_n(F).$$

Since $q(f_n(F)) \to 0$, $q(\partial U_n) \to 0$. From the choice of U_n, it impiles that $q(U_n^c) \to 0$. On the other hand U_n^c contains all but one B_i'. This is possible only if $k = 2$ and if one of the sets B_i', say $B_i' = B_2'$, contains only the point c. By applying Theorem 4.1.10 to f_n^{-1}, we have B_2 contains only a single point and f_n can be extended to a conformal mapping

$$f_n^* : D \cup B_2 \to D' \cup B_2'$$

where $B_2' = \{c\}$. Then $f_n^*(x) \to c$ pointwise in $D \cup B_2$. If B_1 contains more than one point, $\{f_n^* | n \in \mathbb{N}\}$ is a normal family. Thus $f_n^* \to c$ c-uniformly. By Theorem 4.1.13, c is a boundary point of $D' \cup B_2'$. This is a contradiction because $B_2' = \{c\}$.
□

As applications of the previous results we will prove:

Theorem 4.1.16. *Suppose that D and D' are domains, each of which has at least two boundary points. Suppose also that F is a compact subset in D. For $\epsilon > 0$ there is $\delta > 0$ with the following property: If $f : D \to D'$ is a conformal mapping such that $q\left(f(F), \partial D'\right) < \delta$, then $q\left(f(F)\right) < \epsilon$.*

Proof. Suppose that the theorem is not true. Then there is $\epsilon > 0$ and a sequence $f_n : D \to D'$ of conformal mappings such that

$$q\left(f_n(F), \partial D'\right) < \frac{1}{n} \qquad \text{and} \qquad q\left(f_n(F)\right) \geq \epsilon.$$

Since each f_n omits two fixed values, $\{f_n | n \in \mathbb{N}\}$ is a normal family. We may assume that $f_n \to f$ c-uniformly in D. By Theorem 4.1.15, f is either a conformal mapping onto D' or a constant in $\partial D'$. Since $q\left(f_n(F), \partial D'\right) < \frac{1}{n}$,

$$q\left(f_n(F), \partial D'\right) \to 0 \text{ as } n \to \infty.$$

If f is conformal, $q\left(f(F), \partial D'\right) > 0$. But

$$q\left(f_n(F), \partial D'\right) \to q\left(f(F), \partial D'\right) > 0$$

which contradicts that $q\left(f_n(F), \partial D'\right) \to 0$. Hence it is impossible that f is conformal. Since

$$q\left(f_n(F)\right) \geq \epsilon,$$

f cannot be a constant in $\partial D'$. Hence the theorem is true.
□

Theorem 4.1.17. *Suppose that D and D' are domains such that ∂D has at least three points and exactly k components, $2 \leq k < \infty$. Suppose also that F is a compact set in D, consisting of at least two points. Then there exists a positive number δ such that*

$$q\left(f(F), \partial D'\right) > \delta \qquad \text{and} \qquad q\left(f(F)\right) > \delta$$

for every conformal mapping $f : D \to D'$.

Proof. Suppose that the theorem is not true. Then for every $n \in \mathbb{N}$, there is a conformal mapping $f_n : D \to D'$ such that

$$q\left(f_n(F), \partial D'\right) \leq \frac{1}{n} \qquad \text{or} \qquad q\left(f_n(F)\right) \leq \frac{1}{n}.$$

Note that $\partial D'$ has exactly k components, hence every f_n omits two fixed values. By Theorem 3.2.6, $\{f_n | n \in \mathbb{N}\}$ is normal family. We may assume that $f_n \to f$ c-uniformly in D. Since ∂D has at least three points and exactly k components $2 \leq k < \infty$, f is a conformal mapping by Theorem 4.1.15. Because F is a compact set containing at least two points and f is a conformal mapping,

$$q\left(f(F)\right) > 0.$$

Thus there is a subsequence $\{f_{n_k}\}$ such that either

$$q\left(f_{n_k}(F), \partial D'\right) \leq \frac{1}{n_k} \quad \text{for all } k \qquad \text{or} \qquad q\left(f_{n_k}(F)\right) \leq \frac{1}{n_k} \quad \text{for all } k.$$

For the first case,

$$q\left(f_{n_k}(F), \partial D'\right) \to 0 \ \text{ as } \ k \to \infty,$$

which contradicts the fact that

$$q\left(f_{n_k}(F), \partial D'\right) \to q\left(f(F), \partial D'\right) > 0.$$

Similarly, if $q\left(f_{n_k}(F)\right) \to 0$, it contradicts the fact that

$$q\left(f_{n_k}(F)\right) \to q\left(f(F)\right) > 0.$$

Therefore the theorem must be true.

□

Chapter 5

Quasiconformal Mappings

In this section we define the concept of a quasiconformal mapping and observe that the main theorems on normal families, equicontinuity and convergence remain true for quasiconformal mappings.

5.1 The Dilatations of a Homeomorphism

Definition 5.1.1. *Let D and $D^{'}$ be domains in $\overline{\mathbb{R}^2}$ and let $f : D \to D^{'}$ be a homeomorphism. Set*

$$K_I(f) = sup\frac{M(\Gamma^{'})}{M(\Gamma)}, K_O(f) = sup\frac{M(\Gamma)}{M(\Gamma^{'})}, \text{ and } K(f) = max\left\{K_I(f), K_O(f)\right\},$$

*where the suprema are taken over all path families Γ in D such that $M(\Gamma)$ and $M(\Gamma^{'})$ are not simulataneously 0 or ∞. We call $K_I(f), K_O(f)$ and $K(f)$ the **inner**, the **outer** and the **maximal dilatation** of f in D, respectively.*

Remark 5.1.2.

1. *Clearly, $0 \leq K_I(f), K_O(f) \leq \infty$, and either $K_I(f) \geq 1$ or $K_O(f) \geq 1$.*
2. *In fact, it can be shown that both $K_I(f)$ and $K_O(f)$ are at least one.*
3. *If f is conformal, then $M(\Gamma) = M(\Gamma^{'})$ for each path family Γ in D, and, consequently, $K_I(f) = K_O(f) = 1$.*
4. *$M(\Gamma^{'}) \leq K_I(f)M(\Gamma)$ and $M(\Gamma) \leq K_O(f)M(\Gamma^{'})$ for all path families Γ.*

Theorem 5.1.3. *Let $D, D^{'}$ and $D^{''}$ be domains in $\overline{\mathbb{R}^2}$ and let $f : D \to D^{'}$ and $g : D^{'} \to D^{''}$ be homeomorphisms. Then*

(1) *$K_I(f^{-1}) = K_O(f)$.*

(2) *$K_O(f^{-1}) = K_I(f)$.*

(3) $K(f^{-1}) = K(f)$.

(4) $K_I(g \circ f) \leq K_I(g)K_I(f)$.

(5) $K_O(g \circ f) \leq K_O(g)K_O(f)$.

(6) $K(g \circ f) \leq K(g)K(f)$.

Proof. (1) and (2) are clear by Definition 5.1.1.

(3) From (1) and (2) we get

$$K(f^{-1}) = \max\left\{K_I(f^{-1}), K_O(f^{-1})\right\} = \max\{K_O(f), K_I(f)\} = K(f).$$

(4) Let Γ be a path family in D and let $\Gamma^{'} = f(\Gamma), \Gamma^{''} = g(\Gamma^{'})$. Then

$$M(\Gamma^{''}) \leq K_I(g)M(\Gamma^{'}) \leq K_I(g)K_I(f)M(\Gamma),$$

and so

$$\frac{M(\Gamma^{''})}{M(\Gamma)} \leq K_I(g)K_I(f).$$

Taking the supremum we obtain $K_I(g \circ f) \leq K_I(g)K_I(f)$.

(5) The proof is analogous to that of (4).

(6) From (4) and (5) we obtain

$$\begin{aligned} K(g \circ f) &= \max\{K_I(g \circ f), K_O(g \circ f)\} \\ &\leq \max\{K_I(g)K_I(f), K_O(g)K_O(f)\} \\ &\leq \max\{K_I(g), K_O(g)\} \cdot \max\{K_I(f), K_O(f)\} \\ &= K(g)K(f). \end{aligned}$$

□

5.2 Quasiconformal Mappings

Definition 5.2.1. *Let D and $D^{'}$ be domains in $\overline{\mathbb{R}^2}$ and let $f : D \to D^{'}$ be a homeomorphism. If $K(f) \leq K < \infty$, then f is **K-quasiconformal**. Equivalently, f is K-quasiconformal if and only if*

$$\frac{1}{K}M(\Gamma) \leq M(\Gamma^{'}) \leq KM(\Gamma)$$

*for all path families Γ in D. The mapping f is **quasiconformal** if $K(f) < \infty$.*

Remark 5.2.2. *Obviously f is K-quasiconformal if and only if $K_I(f) \leq K$ and $K_O(f) \leq K$.*

As consequences of Theorem 5.1.3 we have the following corollaries.

Corollary 5.2.3. *If f is K-quasiconformal, then f^{-1} is K-quasiconformal.*

Proof. Since $K(f) \leq K < \infty$, we get

$$K(f^{-1}) = K(f) \leq K < \infty.$$

Thus f^{-1} is K-quasiconformal.

□

Corollary 5.2.4. *If $h = g \circ f$, where f is K_1-quasiconformal and g is K_2-quasiconformal, then h is K_1K_2-quasiconformal.*

Proof. Since $K(f) \leq K_1$ and $K(g) \leq K_2$, Theorem 5.1.3 implies that

$$K(h) = K(g \circ f) \leq K(g)K(f) \leq K_1K_2.$$

Therefore h is K_1K_2-quasiconformal as desired.

□

We close this thesis with quasiconformal analogues of Montel's theorem and of the main convergence theorem (Theorem 4.1.4):

Theorem 5.2.5. *Let $\mathscr{F}$ be a family of K-quasiconformal mappings of a domain D into $\overline{\mathbb{R}^2}$. If each $f \in \mathscr{F}$ omits 2 values a_f, b_f with chordal distance*

$$q(a_f, b_f) \geq r,$$

where $r > 0$ is fixed, then $\mathscr{F}$ is equicontinuous and hence a normal family.

Proof. Let $x_0 \in D$ and $0 < \epsilon < r$. Similar to the proof of Theorem 3.1.2, we can choose neighborhoods U and V of x_0, for example disks, so that $\overline{U} \subset V \subset D$, that $A = V \setminus \overline{U}$ is a ring domain, and so that $KM(\Gamma_A) < \lambda(\epsilon)$, where λ is the function introduced in Definition 2.5.1. Then $f(A) = R(C_0, C_1)$, where $C_0 = f(\overline{U})$ and $C_1 = (f(V))^c$, is a ring. Since C_1 contains a_f and b_f, we get

$$q(C_1) \geq q(a_f, b_f) \geq r.$$

For each $x \in U$,

$$q\left(f(x), f(x_0)\right) \leq q(C_0).$$

From the definition of $\lambda(t)$ and K-quasiconformality of f, we obtain

$$KM(\Gamma_A) \geq M(\Gamma_{f(A)}) \geq \lambda(t).$$

where $t = min\left(r, q\left(f(x), f(x_0)\right)\right)$. Thus $\lambda(t) < \lambda(\epsilon)$. Since λ is increasing, we have $t < \epsilon < r$. Hence

$$q\left(f(x), f(x_0)\right) < \epsilon.$$

for all $x \in U$ and $f \in \mathscr{F}$. Therefore $\mathscr{F}$ is equicontinuous at x_0. Finally, $\mathscr{F}$ is normal by Ascoli's theorem.

□

Corollary 5.2.6. *Let $\mathscr{F}$ be a family of K-quasiconformal mappings of a domain D. Then $\mathscr{F}$ is equicontinuous if one of the following conditions is satisfied:*

(1) *There are points $x_1, x_2 \in D$ and a number $r > 0$ such that each $f \in \mathscr{F}$ omits a point a_f and $q(a_f, f(x_i)) \geq r$ for $i = 1, 2$.*

(2) *There are points $x_1, x_2, x_3 \in D$ and a number $r > 0$ such that each $f \in \mathscr{F}$ satisfies the three inequalities $q(f(x_i), f(x_j)) \geq r$, for $i \neq j$.*

Proof. The proof is analogous to the proof of Theorem 3.1.3, except using Theorem 5.2.5 instead of Theorem 3.1.2.

□

In the proof of Lemma 4.1.2, the property of conformality required in the proof is only that every conformal mapping is a homeomorphism. Since a K-quasiconformal mapping is a homeomorphism, Lemma 4.1.2 can be stated in a more general version as following.

Lemma 5.2.7. *Let $f_n : D \to D_n$ be K-quasiconformal, let $f_n \to f$ pointwise in D, and let $\{f_n | n \in \mathbb{N}\}$ be equicontinuous in D. If there are distinct points $z_1, z_2 \in D$ such that $f(z_1) = f(z_2)$, then every neighborhood U of z_1 contains a point $x_0 \neq z_1$ such that $f(z_1) = f(x_0)$.*

Lemma 5.2.8. *Let $f_n : D \to D_n$ be K-quasiconformal, let $f_n \to f$ pointwise in D, and let $\{f_n | n \in \mathbb{N}\}$ be equicontinuous in D. Then every $x_0 \in D$ has a neighborhood U such that $f|U$ is either injective or constant.*

Proof. The proof is analogous to that of Lemma 4.1.3, except in the last paragraph where the conformality is applied. In K-quasiconformal case, we have $M(\Gamma_{A_n}) \leq KM(\Gamma_A)$. Then the sort of the proof of Lemma 4.1.3 is still valid here.

□

By using the same argument in the proof of Theorem 4.1.4, we can prove the following theorem by applying Corollary 5.2.6, Lemma 5.2.7 and Lemma 5.2.8 instead of Theorem 3.1.3, Lemma 4.1.2 and Lemma 4.1.3 respectively.

Theorem 5.2.9. *If $f_n : D \to D_n$ is K-quasiconformal and $f_n \to f$ pointwise in D, then there are 3 possibilities:*

(1) *f is a constant. The convergence may be c-uniform or not.*

(2) *f assumes exactly 2 values, one of which at exactly one point. The convergence is not c-uniform.*

(3) *f is a homeomorphism onto a domain D'. The convergence is c-uniform.*

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Appendix

The following results are from [3].

Let Ω be a set. A nonempty collection $\mathcal{A}$ of subsets of Ω is called a **σ-algebra** if the following two conditions are satisfied:

1. $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$.
2. $\{A_n\}_n \in \mathcal{A}$ implies $\bigcup_n A_n \in \mathcal{A}$.

Let $\mathcal{B}$ be the smallest σ-algebra of subset of $\mathbb{R}^2$ that contains all open sets of $\mathbb{R}^2$. Members of $\mathcal{B}$ are called **two-dimensional Borel sets**.

Let Ω be a set and $\mathcal{A}$ a σ-algebra of subsets of Ω. A **measure**, μ, on $\mathcal{A}$ is an extended real-valued function satisfying the following conditions:

1. $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.
2. $\mu(\phi) = 0$.
3. If $A_1, A_2, \cdots$ are in $\mathcal{A}$, with $A_i \cap A_j = \phi$ for $i \neq j$, then

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

The pair $(\Omega, \mathcal{A})$ is called a **measurable space** and the triple $(\Omega, \mathcal{A}, \mu)$ is called a **measure space**.

A measure on $\mathcal{B}$ is called a **two-dimensional Borel measure**.

We denote by $\widehat{\mathcal{C}}$ the smallest collection of real-valued functions on $\mathbb{R}^2$ that contains the collection of continuous functions and is closed under pointwise limits. The members of $\widehat{\mathcal{C}}$ are called **Borel measurable funcions**. Equivalently, A function f is Borel measurable if and only if the inverse image of each open set under f is a Borel set; that is, f is Borel measurable if and only if $f^{-1}(O) \in \mathcal{B}$ for all open sets O.

Note that a function f is Borel measurable if and only if $\{x \in \mathbb{R}^2 | f(x) < a\}$ is Borel set for all $a \in \mathbb{R}$.

Biography

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